Static Traffic Assignment Problem. A comparison between Beckmann (1956) and Nesterov & de Palma (1998) models.

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Abstract

Since 1956, the Beckmann model is the reference for the static traffic equilibrium problem not only from a theoretical but also from a practical point of view. In 1998, Nesterov & de Palma developed a new model for the static traffic equilibrium problem. In contrast to the Beckmann model, the travel time on an arc is a variable of an optimization problem and it is not determined only by the flow on this arc. Additionally, road capacity constraints are explicitly modeled. The main objective of this work is to provide clarity on model differences both theoretically and practically. First, we consider qualitative differences between both models. In particular, the notion of delays, congestion, the detection of Braess arcs, and the price of anarchy are presented. Secondly, we study quantitative differences in the solutions of both models using large-scale benchmark instances as well as real data. For the Beckmann model, we solve the corresponding optimization problem using the VISUM software. For the Nesterov & de Palma model, we design algorithms based on non-smooth methods developed by Nesterov in 2005.

Keywords

static traffic assignment problem – user equilibrium – social optimum – Beckmann model – price of anarchy – Braess paradox

1 Introduction

1.1 Motivation

Starting with the mass production of automobiles in the beginning of the last century, transport analysts and economists and, later, mathematicians and computer scientists have considered ways of coping with road congestion. From a driver (user) point of view, the highest economical impact of congestion translates into delays. Wardrop, 1952, pointed out in his Second Principle that congestion can only occur if users choose their routes individually to optimize their own utility functions, which is usually the case in transportation networks. Thus the main focus of research has been on ways of understanding and possibly relieving congestion in this setting in which drivers are free to choose their way. From a game-theoretic point of view, a transportation network is considered at equilibrium when all traffic patterns stabilize (and, thus, also the delays) and no driver has any incentive to change his current route (Wardrop's First Principle, Wardrop, 1952). In this case, we say the system is at a *user equilibrium* state (UE). The other side of the spectrum is when there is a central decision maker that assigns routes to drivers. In this case, the goal is to collectively optimize the utilization of the network; when this goal is achieved we say that the system is at a *social optimum* state (SO). The existence and uniqueness of either states is a non-trivial question, but they can be guaranteed in certain cases for some mathematical models.

1.2 Models and Algorithms

Beckmann et al., 1956, were the first to propose and solve a mathematical model to compute both the UE and SO state. Since then, their model has become standard in transportation networks (e.g., Nagurney, 1993, Boyce et al., 2005 and references therein) and nowadays there are several commercial codes to solve it. In this model, the crucial assumption is the existence of a latency function for each road of the network. As more users use a road, its latency grows, thus making it less attractive. Mathematically the problem usually becomes a minimum cost multicommodity flow problem with non-linear but convex objective in which there are no binding constraints among the flows (the natural road capacity constraints are encoded only through the latency functions).

Parallel to the development of algorithms to solve the underlying mathematical optimization problem, extensions of the Beckmann model with additional constraints have been investigated. They aim at being more realistic. A generalized utility function is then considered, where at the same time a latency function and Lagrange multipliers of the additional constraints are used. However, these models have been little studied mainly due to computational issues, see Larsson and Patriksson, 1999, and references therein.

More recently Nesterov and de Palma, 1998, 2000, 2003, developed an alternative model. In this model, the capacity constraints are kept explicitly in the multicommodity flow problem, and, more crucially the First Wardrop Principle is a consequence of the complementary slackness conditions of a convex optimization problem. In contrast to the Beckmann model, the delays are not computed via latency functions, but rather as the Lagrange multipliers of the capacity constraints of a linear multicommodity flow problem. From a numerical point of view, the new primal-dual subgradient techniques developed by Nesterov, 2005, make this model attractive for large scale networks.

1.3 Aim

In this paper we present both models and their main properties. Our aim is to provide clarity in model differences both theoretically and practically. Due to the different assumptions, models cannot be directly compared. Thus, the following measures are considered in order to determine the usefullness of the models.

- The *price of anarchy*, introduced by Koutsoupias and Papadimitriou, 1999, and defined as the ratio between the total utility at UE and at SO, measures how far the UE is from the best possible use of the network.
- The *Braess paradox*, studied by Braess, 1968, describes the counter-intuitive phenomenon occurring when adding more resources to a transportation network, e.g., adding a road or a bridge, deteriorates the quality of a UE. Given the significant cost of adding resources to a transportation network, a good model should be able to predict Braess-paradox type of problems before actually making a physical change to the network.

Moreover, we numerically investigate the set of congested roads as well as the number of routes used per origin-destination pair. On the one hand, we consider small networks where the UE and the SO can be solved with high accuracy for both models using standard solvers (e.g., CPLEX, MOSEK). On the other hand UE and SO are approximately computed for large scale networks. In case of the Beckmann model, we use the software VISUM, 2006. For the Nesterov & de Palma model, an algorithm based on primal-dual subgradient techniques, Nesterov, 2005, was implemented.

After specifying the problem and describing the Nesterov & de Palma model in Section 2, we compare the Beckmann model with the Nesterov & de Palma model in Section 3. In this section, we first recall the settings of the Beckmann model. Then, using benchmark (Bar-Gera, 2007) and real-word (Bundesamt für Raumentwicklung (ARE), 2005) instances, we investigate numerically main differences on the assignment provided by both models. Moreover, the price of anarchy and the Braess paradox are studied both theoretically and numerically. Concluding remarks and future research are presented in Section 4.

2 Problem Statement and Nesterov & de Palma Model

2.1 Static Traffic Assignment Problem

In the following, we define formally the static traffic assignment problem which is address by Beckmann and Nesterov and de Palma.

We consider a traffic network $G = (\mathcal{N}, \mathcal{A})$, where $\mathcal{N}$ is the set of nodes, i.e., intersections or zones, and $\mathcal{A}$ is the set of the arcs, i.e., the roads. Each arc $a \in \mathcal{A}$ has a *capacity*, c_a, i.e., the maximal number of cars that can cross the road a during a given period of time, and a *free travel time*, $\bar{t}_a$, the minimal travel time needed to travel through the road a at maximal allowed speed.

The goal of the static traffic assignment problem is to allocate a given set of drivers with fixed origins and destinations on the network in order to attain a Social Optimum (SO) state or an User Equilibrium (UE) state. At the SO state the total travel time, i.e., the sum of all drivers' travel time is minimized (second Wardrop principle, 1952). At the UE state each driver selects his fastest route (first Wardrop principle, 1952). The current state of a traffic network is specified by a *flow*, $f \in \mathbb{R}^{|\mathcal{A}|}$, i.e., where users are driving, and a *travel time*, $t \in \mathbb{R}^{|\mathcal{A}|}$, i.e., how long it takes to travel through the roads of the network.

The set of fixed origins and destinations is denoted by $\mathcal{OD} \subset \mathcal{N} \times \mathcal{N}$. For each fixed origin-destination pair ($\mathcal{OD}$-pair), $k \in \mathcal{OD}$, $d_k > 0$ corresponds to the number of drivers travelling during a given period of time from the origin of k to its destination. We denote by $h^k \in \mathbb{R}^{|\mathcal{A}|}$ the flow of the $\mathcal{OD}$-pair $k \in \mathcal{OD}$ and thus, $f = \sum_{k \in \mathcal{OD}} h^k$.

Note that we assume the number of drivers using a road to be constant during the considered period of time. Thus, we use this problem for studying the network load only during short specific periods of day, for example peak hours where the average number of drivers using a road can be considered constant.

2.2 Nesterov & de Palma Model

For Nesterov and de Palma, the capacity c_a of a road $a \in \mathcal{A}$ in a traffic network cannot be violated and as far as there is enough capacity on the roads for allocating all drivers, there is no slowdown on the roads. At capacity limit, if the flow of drivers is not well managed, congestion and thus delays on the roads may occur. One can characterize the model as a *fluid* model. Assumption 1 resumes the previous remarks.

Assumption 1. [Nesterov and de Palma, 2000; Nesterov and de Palma, 2003] *Let (f, t) be a traffic assignment. Then, (f, t) satisfy the following conditions:*

- *The total flow f_a on an arc $a \in \mathcal{A}$ never exceeds the capacity c_a of this arc, $f_a \leq c_a$.*
- *Below capacity the travel time t_a on an arc $a \in \mathcal{A}$ is equal to its free travel time $\bar{t}_a$. At capacity limit, it can take any value larger or equal to the free travel time, i.e.,*

$$\begin{aligned} \textit{if } f_a < c_a &\Rightarrow t_a = \bar{t}_a, \\ \textit{if } f_a = c_a &\Rightarrow t_a \geq \bar{t}_a. \end{aligned}$$

The total travel time is defined as $\sum_{a \in \mathcal{A}} f_a t_a$.

Recall that at SO we assume that the drivers are managed by a central organization, which assigns the drivers on the network in order to minimize the total travel time. Thus, even for a road with flow at capacity limit, the travel time does not exceed the free flow travel time. For Nesterov and de Palma, computing a traffic assignment at SO is equivalent to solving the following minimum linear cost multicommodity flow problem,

$$\begin{array}{llllr} \text{(NdP-SO)} & \min_{f,h} & \sum_{a\in\mathcal{A}} f_a \bar{t}_a & & \\ & \text{s.t.} & f_a = \sum_{k\in\mathcal{OD}} h^k_a \le c_a & \forall\, a \in \mathcal{A} & (1) \\ & & Eh^k = \delta_k & \forall\, k \in \mathcal{OD} & (2) \\ & & h^k \ge 0 & \forall\, k \in \mathcal{OD} & (3) \end{array}$$

where E is the node-arc incidence matrix and δ_k is the node demand vector, i.e.,

$$E_{u,a} = \begin{cases} -1 & \text{if } u \text{ is the tail of arc } a, \\ 1 & \text{if } u \text{ is the head of arc } a, \\ 0 & \text{otherwise.} \end{cases} \qquad \delta_{k,u} = \begin{cases} -d_k & \text{if } u \text{ is the origin of } \mathcal{OD}\text{-pair } k, \\ d_k & \text{if } u \text{ is the destination of } \mathcal{OD}\text{-pair } k, \\ 0 & \text{otherwise.} \end{cases}$$

The objective function of NdP-SO corresponds to the total travel time since there are no delays and thus the travel time is equal to the free travel time, $t = \bar{t}$. Equation (1) ensures that capacity constraints are respected. Equations (2) and (3) ensure that the demand of each OD-pair is satisfied, i.e., all drivers have to be correctly assigned. We note that this minimization problem models the SO problem in a very natural manner.

Now let us focus on the capacity constraints (1). From the duality theory the corresponding dual variables are usually considered as the price a user is willing to pay for getting one additional unit of capacity. Nesterov and de Palma interpret it as a penalty, i.e., a delay, that the drivers will face for trying to use a road already at capacity limit. We relax (1), the binding restriction, and consider the corresponding Lagrange Dual problem,

$$\max_{\lambda\ge 0}\ \min_{h_k,\ k\in\mathcal{OD}} \left\{ \langle \bar{t}, \sum_{k\in\mathcal{OD}} h^k \rangle + \langle \lambda, \sum_{k\in\mathcal{OD}} h^k - c \rangle \mid Eh^k = \delta_k,\ h^k \ge 0\ \forall\, k \in \mathcal{OD} \right\}. \tag{4}$$

We observe that for fixed $\bar{\lambda} \ge 0$ and each $k \in \mathcal{OD}$, the minimization problem

$$\min_{h_k} \left\{ \langle \bar{t} + \bar{\lambda}, h^k \rangle - \langle \lambda, c \rangle \mid Eh^k = \delta_k,\ h^k \ge 0 \right\}, \tag{5}$$

is a minimum cost flow problem without capacity constraints, where the cost corresponds to the total travel time for assigning drivers of $\mathcal{OD}$-pair k given the travel time $t = \bar{t} + \bar{\lambda}$. Hence, it is then sufficient to distribute the drivers d_k along a shortest path for the commodity k with respect to the given travel time $t = \bar{t} + \bar{\lambda}$. Having established the duality relation, we observe that for an optimal solution of NdP-SO f^* and optimal Lagrange multipliers λ^* for (1) we have

$$\begin{array}{ll} (f^*, \bar{t}) & \text{is a traffic assignment at SO,} \\ (f^*, \bar{t} + \lambda^*) & \text{is a traffic assignment at UE.} \end{array}$$

It is important to note here that the flow is the same at UE and at SO. UE and SO states only differ in their travel times, i.e., the Lagrange multipliers. Thus, from the point of view of traffic management, λ^* can be used as an incentive for selfish drivers to reach the SO. One may think of the use of toll (road pricing) or a calibration of maximal allowed speeds and/or flow capacities.

Let us consider again the Lagrange Dual problem (4). Denote $\mathcal{P}_k$ the set of all paths between origin and destination of the $\mathcal{OD}$-pair k, and a^k_P the arc incidence vector for each path $P \in \mathcal{P}_k$.

Then, the length of the shortest path for the $\mathcal{OD}$-pair k given the travel time t, $T_k(t)$, is defined by

$$T_k(t) = \min_{P \in \mathcal{P}_k} \{ \langle a_P^k, t \rangle \}. \quad (6)$$

Using (5) and (6), the Lagrange dual of the NdP-SO problem, (4), becomes

$$\max_{t \geq \bar{t}} \{ \sum_{k \in \mathcal{OD}} d_k T_k(t) - \langle t - \bar{t}, c \rangle \},$$

where we replaced $\bar{t} + \lambda$ by t. Theorem 2 resumes the previous remarks.

Theorem 2. [Nesterov and de Palma, 2000; Nesterov and de Palma, 2003]
The arc travel time t^* and the arc flow vector f^* correspond to a traffic assignment at user equilibrium (UE) if and only if t^* is a solution of the following problem

$$\text{(NdP-UE)} \quad \max_{t \geq \bar{t}} \{ \sum_{k \in \mathcal{OD}} d_k T_k(t) - \langle t - \bar{t}, c \rangle \}, \quad (7)$$

and $f^* = c - s^*$, where s^* is a vector of optimal dual multipliers for the inequality constraints $t \geq \bar{t}$ in (7).

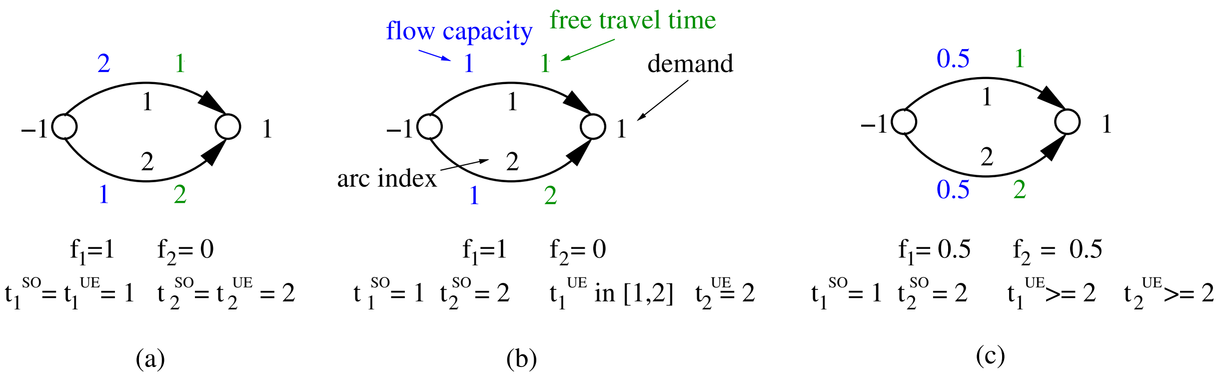


Figure 1: Non uniqueness of delays

The existence and uniqueness of the SO and UE states for the Nesterov & de Palma model is a delicate question. It is obvious that as long as the NdP-SO problem is feasible, the SO state exists but may be not unique. The existence of UE is more restrictive. If the Lagrange multipliers are not unique the delays and, hence the travel times, can not be determined exactly. In this case the mathematical model indicates that the distribution of the drivers on the transportation network is unstable with respect to a small change of the capacity of the roads. In the example of Figure 1 we have in (a) a situation where delays are unique, and actually equal to zero. In (b), since the upper road is used at its capacity limit, the delay on this road is bounded by the difference of the free travel times of both alternative routes. Finally in (c) we have the flow on both roads at capacity limit and thus, the travel time of both roads have to be equal but the delays may be unbounded.

In case of unboundedness of delays, there is at least one $\mathcal{OD}$-pair having no other alternative route, such that any small decrease of flow capacity make the NdP-SO problem infeasible. One can show that it is enough to find one single road $a \in \mathcal{A}$ for which any reduction of capacity makes the NdP-SO problem infeasible for having unbounded delays and vice versa, see Rudyk, 2007.

3 Comparison with the Beckmann Model

In this section, we will first summarize the basic properties of the Beckmann model and then present some comparisons between Nesterov & de Palma and Beckmann model based on numerical results.

3.1 Beckmann Model

In the Beckmann model, one assumes that the travel time for each arc $a \in \mathcal{A}$ only depends on the flow on the arc and it is defined by a latency function $l_a(\cdot)$, which is convex, continuous, nonnegative and nondecreasing. Moreover, the capacity constraints are taken into account by choosing $l_a(\cdot)$ such that a violation of the capacity c_a is penalized. Note that this allows solutions where the capacity constraints are violated. The total travel time of a traffic assignment $(f, l(f))$ is defined by $\sum_{a\in\mathcal{A}} l_a(f_a)f_a$. Under these assumptions, the SO is the solution of the following convex optimization problem

$$\begin{array}{lll}
\text{(B-SO)} \quad \min_{f,h} & \sum_{a\in\mathcal{A}} f_a l_a(f_a) & \\
\text{s.t.} & f_a = \sum_{k\in\mathcal{OD}} h_a^k & \forall\, a \in \mathcal{A} \\
& Eh^k = \delta_k & \forall\, k \in \mathcal{OD} \\
& h^k \geq 0 & \forall\, k \in \mathcal{OD}
\end{array}$$

As in the Nesterov & de Palma model, this problem corresponds to a minimum cost multicommodity flow problem. However, with an objective function that may be non-linear and without capacity constraints. Typically used latency functions are the BPR functions (Bureau of Public Road, 1964) given by

$$l_a(f_a) := \bar{t}_a(1 + \alpha \frac{f_a}{c_a}^{\beta}), \qquad \alpha, \beta \geq 0, \qquad \forall\, a \in \mathcal{A} \tag{8}$$

They penalize more or less the overflow on the roads depending on the values of the parameters α and β, see the example in Figure 2.

Recall that $\mathcal{P}_k$ denotes the set of all paths for the $\mathcal{OD}$-pair k. We denote by $h_P^k \in \mathbb{R}$ the flow of $\mathcal{OD}$-pair k along path $P \in \mathcal{P}_k$ and note that the total flow f_a on road $a \in \mathcal{A}$ can be then computed as follows

$$f_a = \sum_{k\in\mathcal{OD}} \sum_{\{P\in\mathcal{P}_k, a\in P\}} h_P^k.$$

The travel time of a path P given the total flow f is defined by $l_P(f) = \sum_{a\in P} l_a(f_a)$. The first Wardrop principle in this context can be stated as follows:

Principle. [First Wardrop principle, 1952] *The total flow, f^*, satisfies the first Wardrop principle if and only if*

$$\forall\, k \in \mathcal{OD} \text{ and } \forall\, P, Q \in \mathcal{P}_k \text{ such that } {h_P^k}^* > 0 \Rightarrow \sum_{a\in P} l_a(f_a^*) \leq \sum_{a\in Q} l_a(f_a^*), \tag{9}$$

i.e., only the shortest paths are used for each $\mathcal{OD}$-pair.

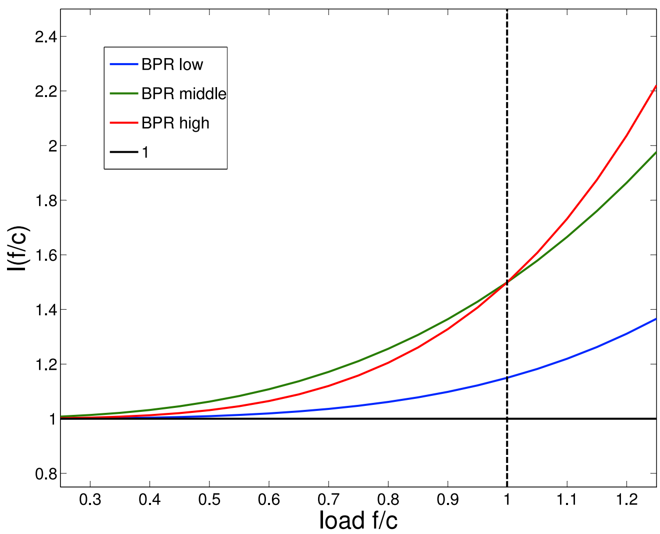


Figure 2: Typical latency functions, BPR, Bureau of Public Road, 1964

One can show that condition (9) corresponds to the optimality conditions of the following convex optimization problem (see Beckmann et al., 1956 and Patriksson, 1994)

$$
\begin{array}{llll}
\text{(B-UE)} & \min_{f,h} & \sum_{a\in\mathcal{A}} \int_0^{f_a} l_a(x)dx & \\
& \text{s.t.} & f_a = \sum_{k\in\mathcal{OD}} h_a^k & \forall\, a \in \mathcal{A} \\
& & Eh^k = \delta_k & \forall\, k \in \mathcal{OD} \\
& & h^k \geq 0 & \forall\, k \in \mathcal{OD}
\end{array}
$$

Each optimal solution f^* of the optimization problem B-UE corresponds to a traffic assignment, $(f^*, l(f^*))$ at UE for the Beckmann model.

The UE and the SO in this setting always exist as long as the set

$$\{(h^k)_{k\in\mathcal{OD}} \mid Eh^k = \delta_k, h^k \geq 0 \quad \forall\, k \in \mathcal{OD}\}$$

is nonempty, since the B-SO and B-UE problems are convex optimization problems. The literature on methods to solve the B-SO and the B-UE is large. See for an overview the paper of Boyce et al., 2005. In particular, Bar-Gera, 2002, develops an origin-based algorithm specially efficient for solving the UE problem when highly accurate solutions for small and large-scale instances are required.

Extensions of the Beckmann model, where additional constraints are considered, have also been investigated, Larsson and Patriksson, 1999. However, they have not been deeply studied. Mathematically, the UE with additional constraints is formulated as follows

$$\text{(B-UEext)} \quad \min_{f,h} \quad \sum_{a\in\mathcal{A}} \int_0^{f_a} l_a(x)dx$$
$$\text{s.t.} \quad g_i(f) \leq 0 \qquad \forall i \in \mathcal{I} \quad \text{(additional constraints)}$$
$$f_a = \sum_{k\in\mathcal{OD}} h_a^k \qquad \forall\, a \in \mathcal{A}$$
$$Eh^k = \delta_k \qquad \forall\, k \in \mathcal{OD}$$
$$h^k \geq 0 \qquad \forall\, k \in \mathcal{OD},$$

where $\mathcal{I}$ is a subset of indices of arcs, nodes or $\mathcal{OD}$ pairs, and $g_i(f)$ are additional constraints which are assumed to be convex and continuous differential functions. Again, the first Wardrop principle corresponds to the optimality conditions of the convex optimization problem B-UEext. The path's travel time are in this setting generalized as follows

$$t_P(f^*, \zeta^*) = \sum_{a\in P} l_a(f_a^*) + \sum_{i\in\mathcal{I}} \zeta_i^* \left(\sum_{a\in P} \frac{\partial g_i(f^*)}{\partial f_a} \right), \quad \forall\, P \in \mathcal{P}_k,\, \forall\, k \in \mathcal{OD},$$

where f^* is an optimal solution of B-UEext and ζ^* are the Lagrange multipliers corresponding to the additional constraints. Consider for example the special case of flow capacity constraints on the roads, $g_a(f) := f_a - c_a$. For an optimal solution f^* and the optimal Lagrange multipliers ζ^*, the generalized travel time for each road $a \in \mathcal{A}$ is given by $t_a(f_a^*) = l_a(f_a^*) + \zeta_a^*$. This corresponds to the Nesterov & de Palma model in the case of constant latency functions, $l_a(f_a) = \bar{t}_a \;\forall\, a \in \mathcal{A}$.

B-UEext is still a convex optimization problem. Thus, a solution exists as long as the set

$$\{(h^k)_{k\in\mathcal{OD}} \mid g_i(f) < 0 \;\forall i \in \mathcal{I},\; Eh^k = \delta_k, h^k \geq 0 \quad \forall\, k \in \mathcal{OD}\}$$

is not empty. However, B-UEext is computationally more difficult to solve since the additional constraints are often binding constraints. Moreover, the existence of a UE is a non trivial question as in Nesterov & de Palma model. The non uniqueness of the dual multipliers makes that the travel times cannot be exactly determined.

3.2 Numerical Comparison Based on Small-Scale Networks

The models base the travel times on different assumptions. Therefore, a direct comparison of the travel times is not suitable. Instead, we study the influence of the models' assumptions on the drivers' distribution on the network. In particular, we consider the set of congested roads and the number of paths used per $\mathcal{OD}$-pair. In this subsection, we investigate two small instances of the static traffic assignment problem, namely the Sioux Falls and the Anaheim networks, which can be solved with high accuracy (10^{-6}) using standard solvers. The main characteristics of these two instances are shown in Table 1.

For the Beckmann model, three different BPR functions are used, BPR low, middle, and, high (see Equation (8) and Figure 2). For the extended Beckmann model, we choose the capacity constraints for the roads as additional constraints.

First we investigate the set of congested roads at SO and at UE for both models. In Figures 3 and 4 these sets are depicted for the Sioux Falls network. The roads at capacity limit are drawn

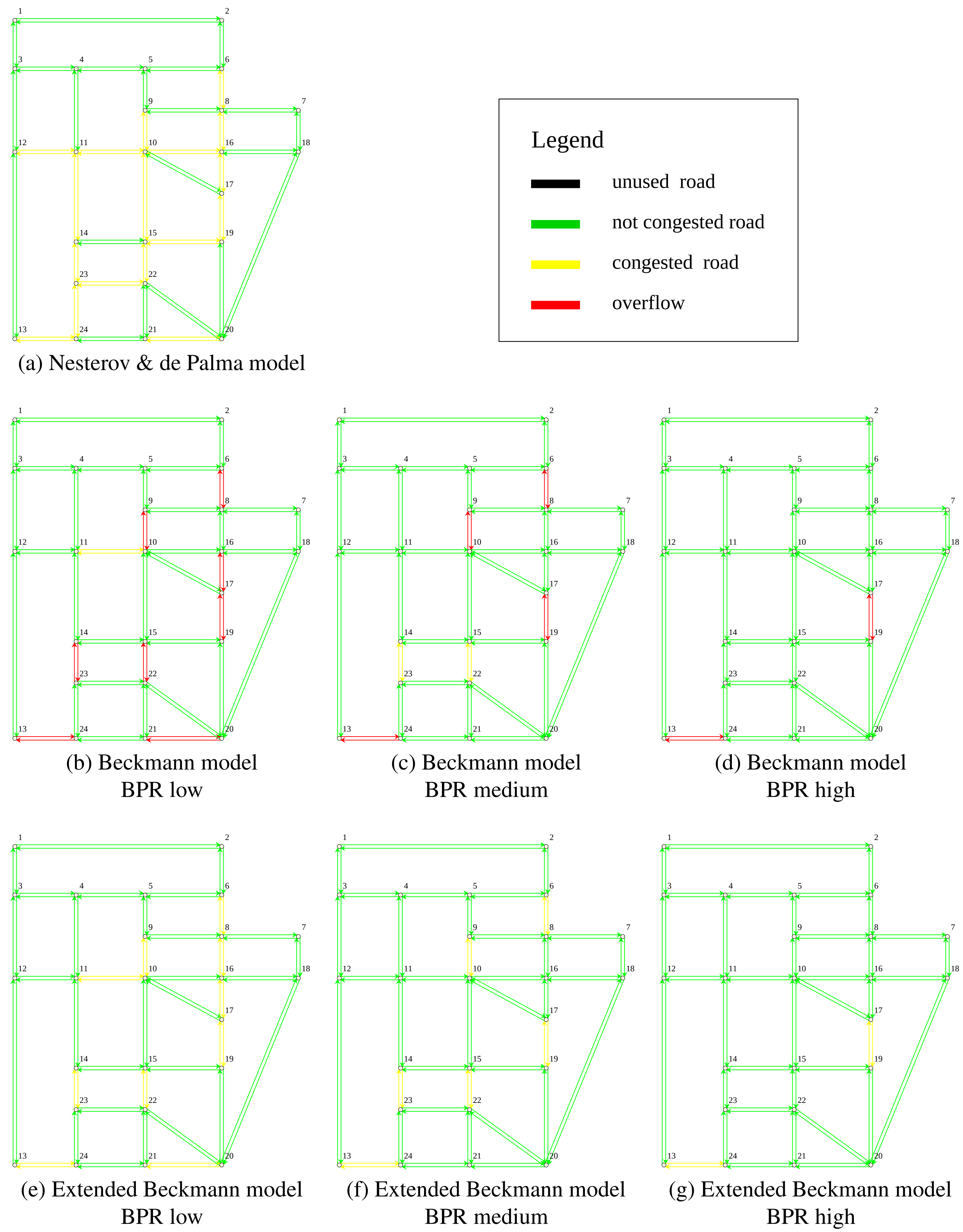
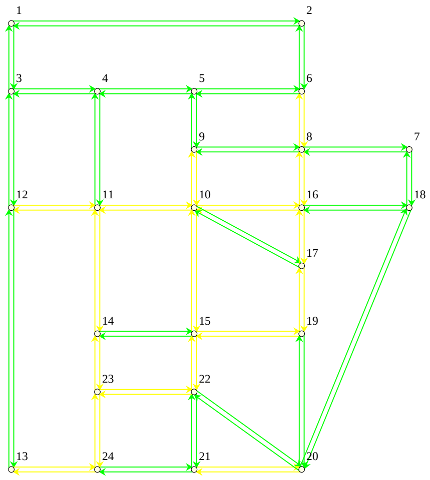
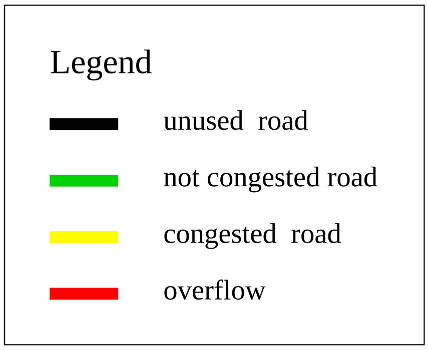
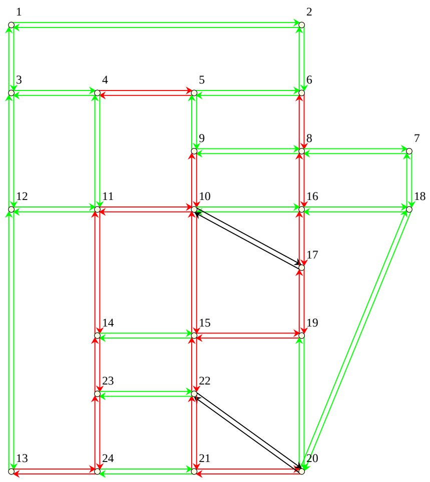


Figure 3: Sioux Falls - flow distribution at Social Optimum

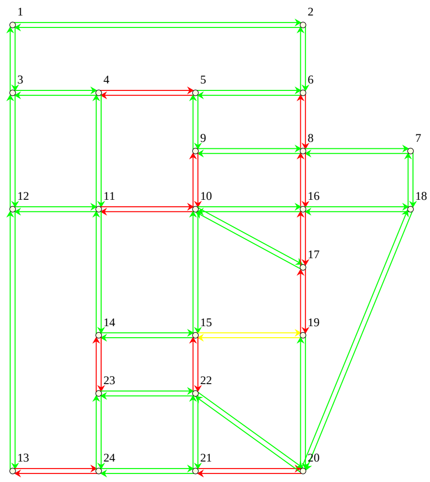


(a) Nesterov & de Palma model

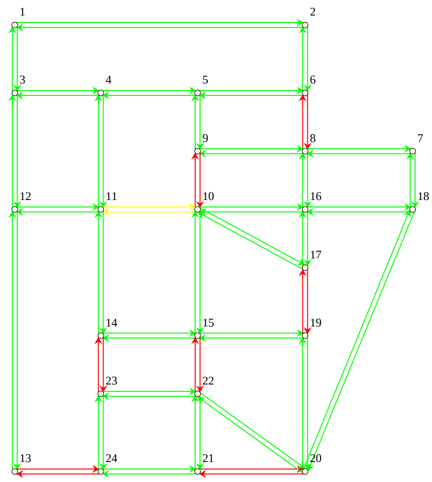




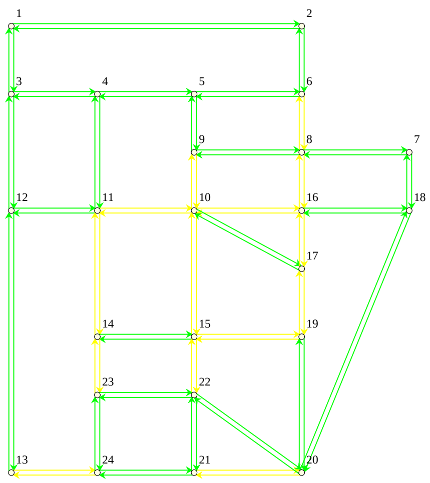
(b) Beckmann model
BPR low



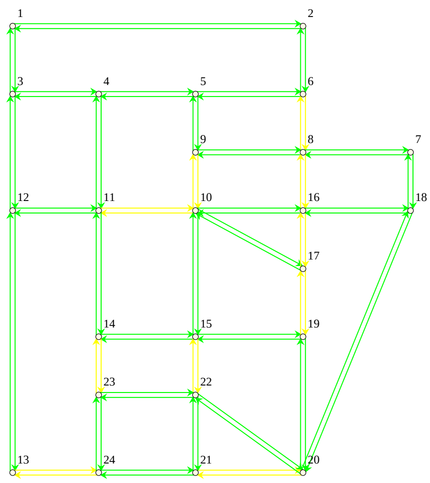
(c) Beckmann model
BPR medium



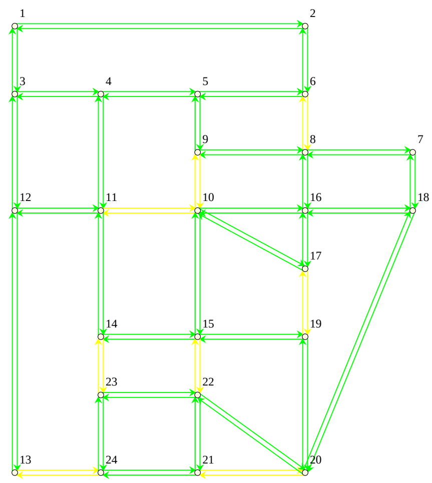
(d) Beckmann model
BPR high



(e) Extended Beckmann model
BPR low



(f) Extended Beckmann model
BPR medium



(g) Extended Beckmann model
BPR high

Figure 4: Sioux Falls - flow distribution at User Equilibrium

Instance	Zones	Nodes	Roads	$\mathcal{OD}$-pairs
Sioux Falls	24	24	76	528
Anaheim	38	416	914	1’405

Table 1: Characteristics of the small networks

in yellow and the roads with overflow are drawn in red. We observe that the set of congested roads provided by the solution of the Nesterov & de Palma model contains the set of congested roads provided by the solutions of the considered Beckmann and extended Beckmann models at SO as well as at UE. The same results can be observed for the Anaheim network. Moreover, as expected, the flow distribution delivered by the solutions of the extended Beckmann model is close to the flow distribution obtained by using the Nesterov & de Palma model. Not surprisingly, if the latency function increases the penalty on the overflow, the overflow in the derived traffic assignment is reduced (see Tables 2 and 3). However, it is interesting to remark that the latency function which best duplicates the solution provided by the Nesterov & de Palma model, both at SO and at UE, is the BPR function with the standard parameters’ values, $\alpha = 0.15$ and $\beta = 4$. These values were defined by the Bureau of Public Road, 1964.

Model		SO		UE	
		average (%)	std (%)	average (%)	std (%)
Beckmann	BPR low	108.5	2.1	116.4	2.8
	BPR medium	104	1.8	111.2	1.9
	BPR high	102.7	4	109	2.1

Table 2: Sioux Falls - average overflow at SO and at UE

Model		SO		UE	
		average (%)	std (%)	average (%)	std (%)
Beckmann	BPR low	106.7	5.6	122.7	9.7
	BPR medium	0	0	110	7.9
	BPR high	0	0	103.9	7.7

Table 3: Anaheim - average overflow at SO and at UE

In Tables 4 and 5, the average numbers of paths used per $\mathcal{OD}$-pair at SO and at UE by both models is summarized. We remark that the flow distribution derived from the Nesterov & de Palma model ($\simeq$ 1 path/$\mathcal{OD}$-pair) uses less paths than the flow distribution derived by the Beckmann models ($\simeq$ 1.4 paths/$\mathcal{OD}$-pair). The addition of capacity constraints on the Beckmann model does not significantly influence the number of paths used.

From our numerical results, we observe that the Nesterov & de Palma model generates traffic assignments where the flow distribution is concentrated as much as possible, whether we look for a SO or a UE state. In contrast, but as expected from the Beckmann model, the more we penalize the overflow the more the flow is spread out over the network.

Model		SO	UE
Nesterov & de Palma		1.057	1.057
Beckmann	BPR low	1.572	1.299
	BPR medium	1.458	1.439
	BPR high	1.545	1.574
Extended Beckmann	BPR low	1.598	1.598
	BPR medium	1.485	1.652
	BPR high	1.545	1.598

Table 4: Sioux Falls - average number of used paths at SO and at UE

Model		S0	UE
Nesterov & de Palma		1.004	1.004
Beckmann	BPR low	1.357	1.277
	BPR medium	1.443	1.398
	BPR high	1.458	1.440
Extended Beckmann	BPR low	1.322	1.289
	BPR medium	1.443	1.403
	BPR high	1.458	1.436

Table 5: Anaheim - average number of used paths at SO and at UE

Price of Anarchy

As already mentioned in the introduction, the price of anarchy was first introduced by Koutsoupias and Papadimitriou, 1999, and it is defined as the ratio between the total utility at UE and at SO. In our context, the total utility corresponds to the total travel time of a traffic assignment (f, t) and is denoted by $U(f,t) = \sum_{a \in \mathcal{A}} f_a t_a$. The price of anarchy is then formulated as follows

$$\text{price of anarchy} = \frac{U(f^{UE}, t^{UE})}{U(f^{SO}, t^{SO})},$$

where (f^{UE}, t^{UE}) corresponds to a traffic assignment at UE and (f^{SO}, t^{SO}) to a traffic assignment at SO. In our context, an upper bound on the price of anarchy is a relative measure on how far a UE is off from a best possible network utilization (SO).

The existence of such bounds for the Beckmann model without additional constraints has been intensively investigated in recent years, see Roughgarden and Tardos, 2000, Roughgarden and Tardos, 2004, Correa et al., 2004 and references therein. For example, Roughgarden, 2003, shows that the price of anarchy is bounded by $O(\frac{d}{\log d})$ when the latency function is a polynomial with nonnegative coefficients and degree at most d. Thus, the price of anarchy for the BPR functions is bounded by $O(\frac{\beta}{\log \beta})$ $(= 2$ for $\beta = 4)$.

For the Nesterov & de Palma model as well as for the Beckmann model with additional constraints, the existence of such a bound is intrinsically related to the boundedness of the delays, i.e., the Lagrange dual multipliers. We reconsider now the example in Figure 1 (c). Using the Nesterov & de Palma model, we get a total travel time at SO of $\frac{3}{2}$. The total travel time at

UE is unbounded since any solution distributing half of the flow on each arc with the delays $\lambda_1 = \lambda_2 + 1$, $\lambda_2 \geq 0$, is a traffic assignment at UE. The price of anarchy is then $\frac{2}{3}(1 + \lambda_1)$. The same observation can be made for the extended Beckmann model.

Each computed price of anarchy gives an information on the utilization of the network. We compute the price of anarchy for the traffic assignments for the Sioux Falls and the Anaheim networks. Table 6 summarizes these results. We observe that the Nesterov & de Palma model is more pessimistic than the Beckmann model, even if capacity constraints are explicitly considered. The difference for the Anaheim instance is much smaller compared to the Sioux Falls instance, since the first does not correspond to a highly loaded network. In the Anaheim instance, only 0.76 % of the roads are at capacity limit for the Nesterov & de Palma model and 0.66 % for the pessimistic UE given by the Beckmann model.

Model		Sioux Falls	Anaheim
Nesterov & de Palma		1.38	1.008
Beckmann	BPR low	1.026	1.002
	BPR medium	1.039	1.006
	BPR high	1.053	1.007
Extended Beckmann	BPR low	1.003	1.002
	BPR medium	1.016	1.006
	BPR high	1.025	1.008

Table 6: Price of Anarchy

Braess Paradox

As already mentioned in the introduction, the Braess paradox occurs when adding more resources to a transportation network, for example adding a road or a bridge, deteriorates the quality of a UE. In other words, more resources create worse delays for the drivers. Braess, 1968, was the first to point out this counter-intuitive fact by exhibiting a simple example using the Beckmann model. This phenomenon can also be interpreted as follows. Suppose we close a road or we increase its free travel time by decreasing the maximal allowed speed in this road. If the utility, i.e., the total travel time, at UE decreases then we observe also a Braess paradox.

In the following we investigate numerically the detection of Braess phenomena on the Sioux Falls network by the three models. For this sake we increase the free travel time for one road at the time and we look for an improvement of the total travel time at UE. In Table 7 the relative improvement of the total travel time is given for the tested roads.

We note that the extended Beckmann model does not detect any Braess paradox. Road 25 and 26 are seen as Braess roads by the Nesterov & de Palma model as well as by the Beckmann model but only for latency function BPR medium and high. Road 15 and 16 are only considered as Braess roads by the Nesterov & de Palma model. The same observation can be made for the Anaheim network.

The Braess paradox is intrinsically tied to the demands of the $\mathcal{OD}$-pairs. After reducing the demands of all $\mathcal{OD}$-pairs by 30 %, neither the Nesterov & de Palma model nor the Beckmann model detects a Braess paradox. From our numerical results, we note that the detection of

Road	Nesterov & de Palma	Beckmann			Extend Beckmann		
		BPR low	BPR medium	BPR high	BPR low	BPR medium	BPR high
15	2.02	-	-	-	-	-	-
16	2.02	-	-	-	-	-	-
25	3.52	-	1.19	1.36	-	-	-
26	3.52	-	1.19	1.36	-	-	-

Table 7: Sioux Falls - Braess Paradox

the Braess Paradox for the Beckmann model depends as expected on the choice of the latency function.

3.3 Numerical Comparison Based on Large-Scale Networks

In the previous subsection, the instances considered were small enough to be solved with very high accuracy by standard solvers (e.g. CPLEX, MOSEK). Now we focus on large scale instances arising from real world problems, which cannot be handled by these solvers anymore, mainly due to memory requirements. Therefore, we compare the traffic assignments at UE generated by both models using approximate methods requiring less computational effort.

For the Beckmann model, we concentrate on the model without additional constraints, B-UE, which is most used in practice, and apply the software VISUM, 2006. The method used by this software is divided into two phases. In the first phase, successive shortest path assignments are made. At each iteration, a part of each $\mathcal{OD}$-pair's demand is assignend to one of its current shortest paths and then the travel time of each arc is updated. In the second phase, the flow for each $\mathcal{OD}$-pair is balanced among the already assigned paths until the travel time of each path is approximately the same, i.e., until equilibrium.

For the Nesterov & de Palma model, we implement an algorithm based on the primal-dual subgradient techniques developed by Nesterov, 2005. We apply these techniques to the NdP-UE problem, i.e., we approximate the travel time t. Contrary to traditional subgradient methods, the information given by the subgradients is kept during all iterations. In our context, this corresponds to the shortest paths for each $\mathcal{OD}$-pair given the current travel time approximation and thus the paths where the drivers will be assignend by the algorithm's iterations.

As already mentioned, the instances that we are going to study arise from real world problems. We consider an area around Zurich, Switzerland, such that each $\mathcal{OD}$-pair can be reached within one hour approximately. Then, we choose different one hour periods during the day to have different demands. Table 8 summarizes the main characteristics of Zurich Regional. These data as well as the parameter values of the BPR latency function were provided by Bundesamt für Raumentwicklung (ARE), 2005. In Figure 5 the devision of the Zurich Regional network in zones is depicted.

Since the software VISUM cannot compute SO states, we focus in this subsection on the study of the UE. The stopping criterion of the method in VISUM is the relative difference of the travel times of the paths used per $\mathcal{OD}$-pair. We set it to 5%. For the primal-dual subgradient algorithm, we choose as stopping criteria a relative gap of 0.005 to optimal solution of the NdP-UE problem.

	Zones	Nodes	Roads
Zurich Regional	784	7'009	16'936

	$\mathcal{OD}$-pairs	Total Demand
00:00 - 01:00	253'705	7'559.74
07:00 - 08:00	433'504	231'255.02
08:00 - 09:00	437'803	127'922.89
12:00 - 13:00	369'449	176'222.85
17:00 - 18:00	443'622	252'871.96
18:00 - 19:00	439'660	175'669.58

Table 8: Characteristics of Zurich Regional network, Bundesamt für Raumentwicklung (ARE)

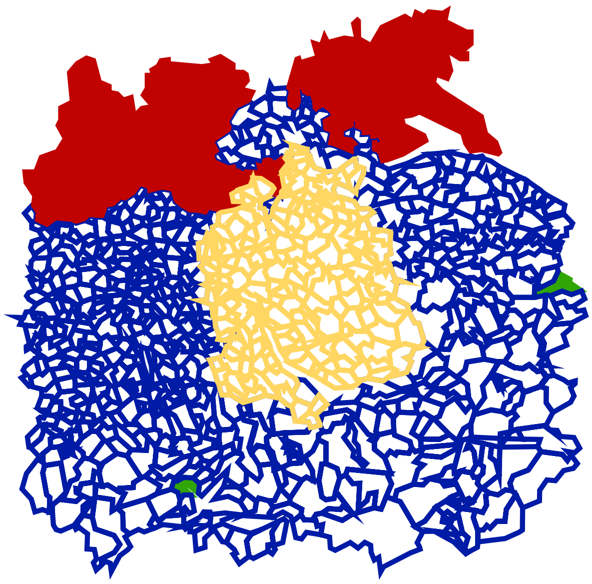


Figure 5: Zurich Regional - zones

In Figures 6 and 7 we observe that the set of congested arcs at the Nesterov & de Palma UE contains the set of congested arcs at the Beckmann UE. However, Table 9 shows that the average congestion (overflow) is higher for the Beckmann traffic assignment than for the Nesterov & de Palma traffic assignment. Recall that overflow cannot happen in the Nesterov & de Palma model. The overflow observed in Table 9 is due to the fact that we are approximately computing the traffic assignment at UE.

	Nesterov & de Palma		Beckmann	
	Number of Congested Arcs (%)	Average Congestion (%)	Number of Congested Arcs (%)	Average Congestion (%)
00:00 - 01:00	0	0	0	0
07:00 - 08:00	1.25	102.65	0.74	114.31
08:00 - 09:00	0.12	100.65	0.01	104.76
12:00 - 13:00	0.08	100.74	0.02	105.68
17:00 - 18:00	0.89	102.17	0.41	112.23
18:00 - 19:00	0.27	100.75	0.05	103.84

Table 9: Zurich Regional - number of congested arcs and average congestion (overflow) at UE

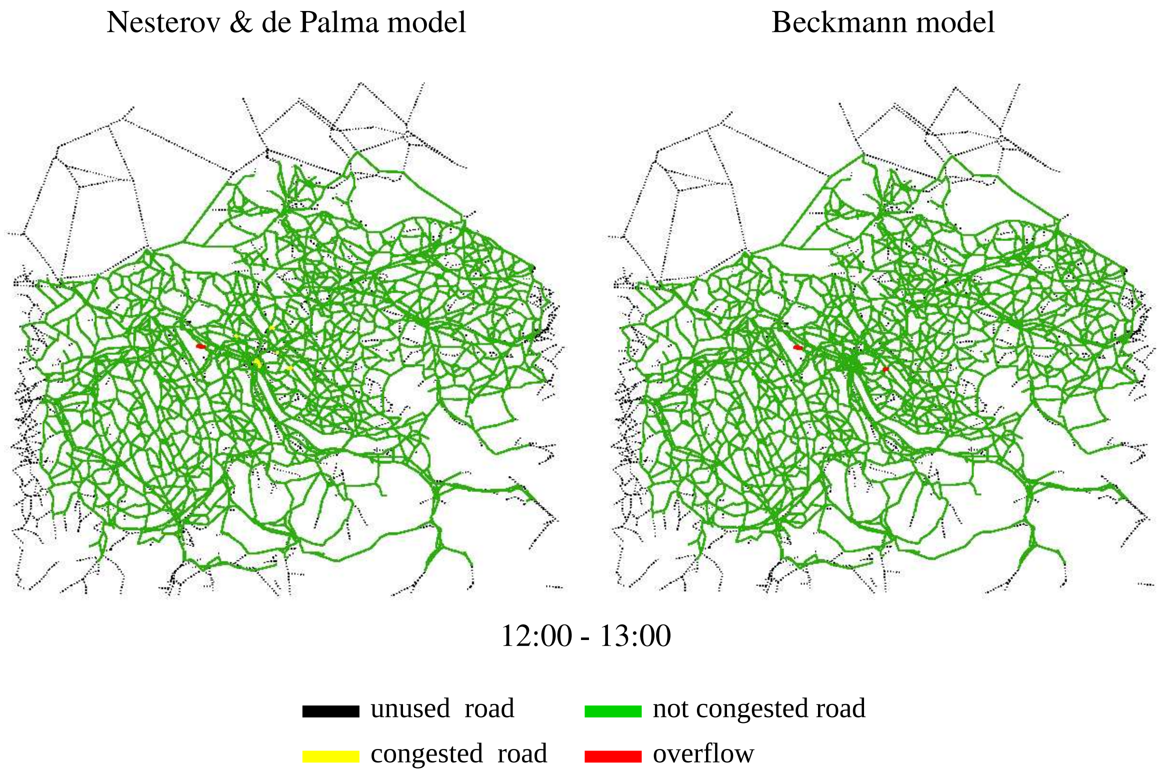


Figure 6: Zurich Regional - flow distribution at User Equilibrium

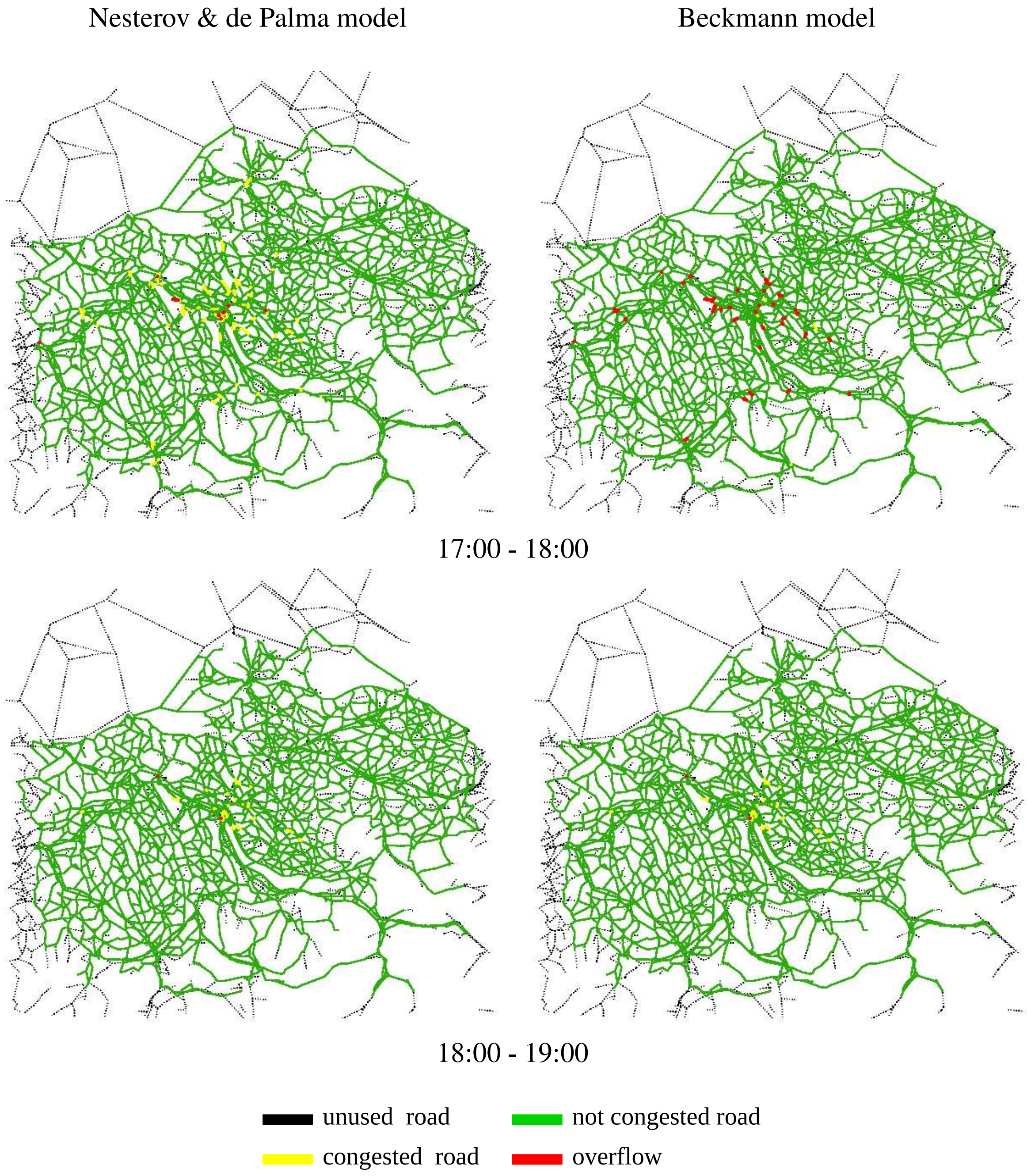


Figure 7: Zurich Regional - flow distribution at User Equilibrium

Figure 8 shows the difference in the distribution of the total flow. We remark that the fraction of arcs with more flow in the Nesterov & de Palma assignment (in orange) is bigger than the fraction of arcs with more flow in the Beckmann assignment (in red).

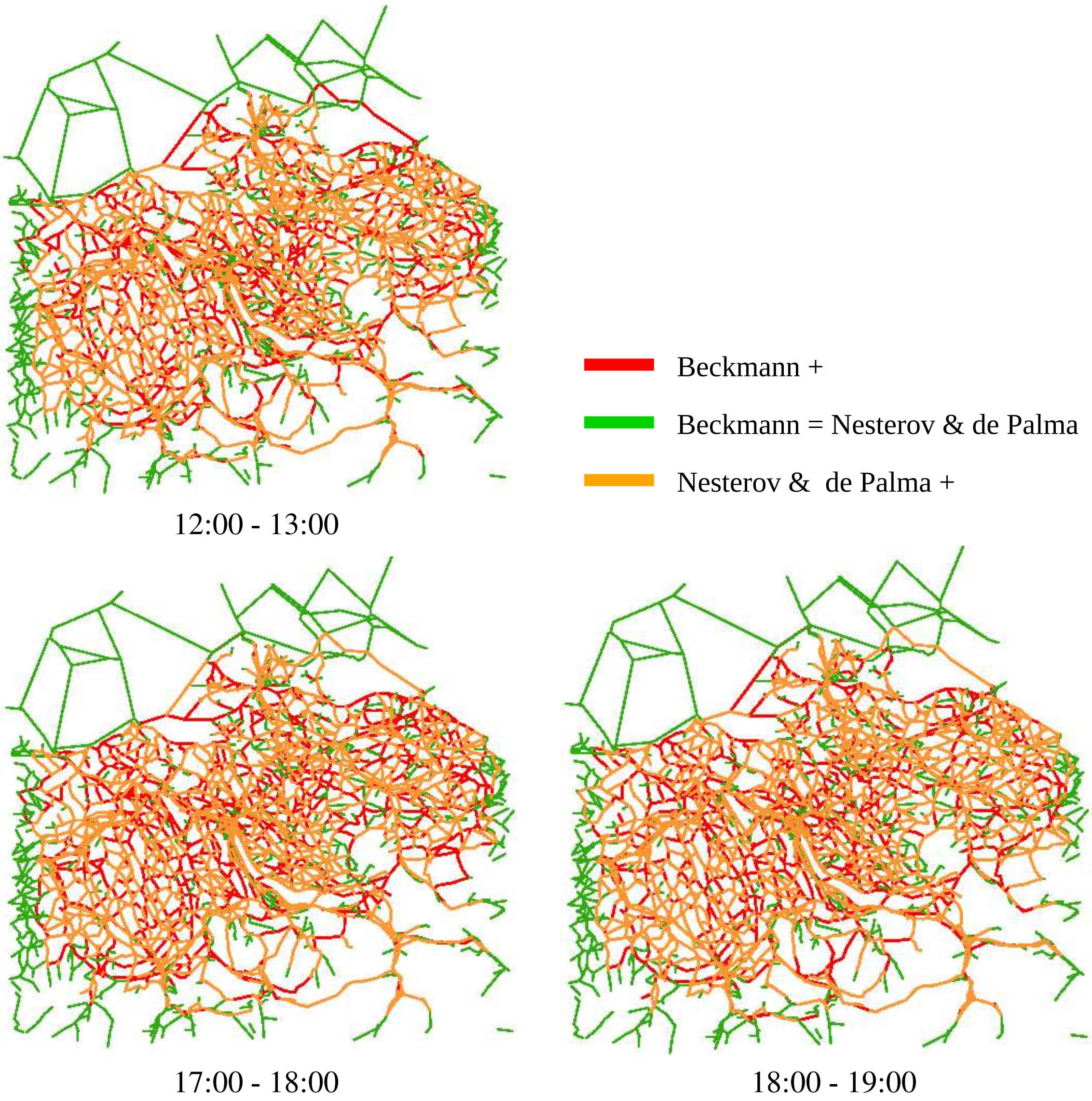


Figure 8: Zurich Regional - flow difference at User Equilibrium

	Nesterov & de Palma		Beckmann	
	Average Number of Paths per $\mathcal{OD}$-pair	Maximal Relative Travel Time Difference	Average Number of Paths per $\mathcal{OD}$-pair	Maximal Relative Travel Time Difference
00:00 - 01:00	1	0	1.076	< 0.05
07:00 - 08:00	13.299	0.24	1.401	< 0.05
08:00 - 09:00	1.137	0.06	1.207	< 0.05
12:00 - 13:00	1.057	0.04	1.162	< 0.05
17:00 - 18:00	2.138	0.15	1.329	< 0.05
18:00 - 19:00	1.320	0.06	1.264	< 0.05

Table 10: Zurich Regional - average number of paths used at UE and maximal relative difference of their travel times

Finally, we consider the average number of paths used per $\mathcal{OD}$-pair and the quality of the corresponding UE, i.e., the accuracy of the travel times of the used paths, see Table 10. For the Beckmann assignment we know that the relative difference in the travel times of the used paths is at most 5% (stopping criterion). For the Nesterov & de Palma model, the corresponding values are shown in the table. Most are larger than 5%. Considering the average number of paths used at UE, we observe that for instances with low $\mathcal{OD}$-pair demands (00:00 - 01:00, 12:00 -13:00), the drivers are less spread out in the Nesterov & de Palma solution than in the Beckmann solution. However, for instances with high $\mathcal{OD}$-pair demands this is not the case. This is mainly due to the applied method. In the primal-dual subgradient method we keep all the information of the shortest paths computed at each iteration of the algorithm and distribute the drivers on these paths in the end. The instances with small $\mathcal{OD}$-pair demands only need a few iterations to reach a solution with high accuracy (relative gap).

4 Summary and Outlook

We compared a new modeling approach for the static traffic assignment problem, the Nesterov & de Palma model, with the well-established Beckmann model, and an extension of the Beckmann model.

The existence of a social optimum (SO) is ensured for both models under minimal requirements. The existence of a user equilibrium (UE) is also easily ensured for the Beckmann model, but it is more restrictive for the Nesterov & de Palma and the extended Beckmann model. The latter two models use minimum cost multicommodity flow problems with capacity constraints and the corresponding Lagrange dual multipliers for defining a UE. Therefore, the existence of a UE and its uniqueness are intrinsically tied to the existence of the Lagrange dual multipliers.

In the Nesterov & de Palma model, duality theory yields that the flow patterns are equal at SO and at UE and that the travel times differ exactly by the Lagrange dual multipliers. This duality relation provides a natural way to offer an incentive to selfish drivers to reach the SO. On the other hand, with the Beckmann and the extended Beckmann model, traffic managers need to adjust the parameters of the latency function to achieve the same result.

As the travel times of the models are based on different assumptions, we did not compare them directly. Instead we focused on a computational comparison of the flow distribution at UE and at SO generated by the models. Both for small and large scale instances we observed that the set of congested roads in the Nesterov & de Palma model includes the set of congested roads in the Beckmann model. For the small instances the drivers are less spread out in the Nesterov & de Palma model than in the Beckmann model. For the large instances with high total demand the drivers are more spread out in the Nesterov & de Palma model than in the Beckmann model.

At the moment our results do not enable us to decide which model better predicts the real traffic flow. In order to answer this question we need to do a comparison with real data from traffic counters. A comprehensive investigation of the extended Beckmann model, using large scale instances, should also be done to clarify the difference to the Nesterov & de Palma model.

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