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Abstract

The integration of discrete choice models with Mixed Integer Linear Programming (MILP) models provides a better understanding of customers' preferences to operators while planning for their systems. However, the formulations associated with the former are highly nonlinear and non convex. To overcome this limitation, we propose a linear formulation of a general discrete choice model that can be embedded in any MILP model by relying on simulation. We characterize a demand-based benefit maximization problem to illustrate the use of this approach. Despite the clear advantages of this integration, the size of the resulting formulation is high, which makes it computationally expensive. Given its underlying structure, we use Lagrangian relaxation to decompose it into two separable subproblems: one concerning the decisions of the operator, that can be written as a Capacitated Facility Location Problem (CFLP), and the other the choices of the customers, for which we need to develop additional strategies to decompose it along the two dimensions that, by design, decompose the problem (the customers and the draws). Finally, we consider a subgradient method to optimize the Lagrangian dual.

Keywords

discrete choice models, mixed integer optimization, simulation, Lagrangian relaxation

1 Introduction

The integration of discrete choice models with Mixed Integer Linear Programming (MILP) models provides a better understanding of the preferences of the customers to the operators while planning for their systems. Despite its clear advantages, the formulations associated with choice models are highly nonlinear and non convex, and therefore difficult to include in MILP.

In Pacheco *et al.* (2017), we overcome this limitation by defining a general framework that allows to characterize almost any discrete choice model based on the random utility principle with a linear set of constraints that can be embedded in any MILP formulation. The probabilistic nature of the choice model is addressed with simulation. For each error term in the utility function, we draw from its distribution.

This approach can be used to model numerous applications, such as the design of a train timetable in transportation or the shelf space allocation problem in retail. For the sake of illustration, we characterize a demand-based benefit maximization problem, where an operator selling services to a market, each of them at a certain price and with a certain capacity (both to be decided), aims at maximizing its benefit (difference between the generated revenues and the operating costs).

We consider a case study from the recent literature to deal with the integration of an existing choice model in the demand-based benefit maximization problem, and to test the extent of the resulting formulation. The obtained results show that this approach is a powerful tool to configure systems based on the heterogeneous behavior of customers. However, the disaggregate representation of customers' preferences and the linearity of the formulation imply that the dimension of the problem is high. Hence, solving it exactly is computationally expensive.

Decomposition techniques are convenient here to speed up the solution approach, and represent an alternative to valid inequalities since they can be applied in a general way. The problem, by design, can be decomposed along two dimensions: the draws, which represent independent behavioral scenarios, and the customers, who individually face an optimization problem to maximize their utility. Furthermore, we can distinguish the decisions to be made by the two agents involved in the problem: the operator and the customers. We rely on Lagrangian relaxation in order to define separable subproblems that can be solved more easily than the original one.

The remainder of the paper is organized as follows. Section 2 reviews the integration of choice and optimization models and the applications of Lagrangian relaxation. Section 3 describes the demand-based benefit maximization problem. In Section 4, we illustrate the decomposition by customers and draws, and in Section 5 we analytically describe the subproblems associated with each agent. Finally, the conclusions and future research are discussed in Section 6.

2 Literature review

The integration of choice models into optimization problems is an increasing trend. They allow to account for demand heterogeneity, as well as other features, such as complex substitution patterns. Several works can be found in many different applications: Haase and Müller (2013) in facility location for schools, Talluri and Van Ryzin (2004) in the field of revenue management and Gilbert *et al.* (2014) in the context of transportation networks, to cite a few.

In the literature, the probabilistic representation of the choice is either included in a deterministic way (the utility is exogenous to the optimization model), or the decision variables of the optimization problem appear in the utility function (endogenous utility). The latter is more challenging because it leads to nonlinear and non convex formulations, but captures the interaction between both models. Moreover, even if inappropriate in reality, various authors place simplistic assumptions on the choice model to come up with tractable and efficient solutions.

In Pacheco *et al.* (2017), we propose a mathematical formulation integrating almost any discrete choice model into MILP. The formulation is linear, to ensure the tractability of the optimization model, and remains fairly general, as it can be used with practically any choice model and any MILP model. However, as mentioned in Section 1, the resulting problem is computationally expensive, and might fail when solving exactly instances with a really high number of customers or when considering a large number of draws to be as accurate as possible.

Lagrangian relaxation (Fisher, 2004) is a technique that exploits the fact that many difficult MILP problems can be seen as relatively easy problems complicated by a set of side constraints. These constraints can be transferred to the objective function with associated parameters (Lagrangian multipliers), which impose a penalty on violations. The relaxed problem provides an upper bound (for a maximization problem) on the optimal value of the original problem. To obtain the tightest bound, a problem on the Lagrangian multipliers (Lagrangian dual) needs to be solved. This problem is typically solved with an iterative method called subgradient method, which uses subgradients of the objective function to update the Lagrangian multipliers at each iteration.

In this research, we consider a variant of Lagrangian relaxation, known as Lagrangian decomposition (Guignard and Kim, 1987). It consists in creating identical copies of some decision variables, and using each copy in each of the set of constraints that enables to decompose the problem into several subproblems. This technique is really interesting as it keeps all the original constraints, and it might lead to a stronger bound than conventional Lagrangian relaxation. For instance, Ertogral (2008) models the integration of inventory and transportation decisions and uses Lagrangian decomposition to characterize a subproblem associated with each decision.

3 Demand-based benefit maximization

In this section, we summarize the main concepts and notations of the demand-benefit maximization problem. We refer the reader to Pacheco *et al.* (2017) for further details. We start by describing the way in which the choice model is linearized to be included in a MILP program. To illustrate our approach, we characterize a MILP formulation modeling a benefit maximization problem. We finally provide some relevant results from the performed case study, which motivate the usage of decomposition techniques to reduce the computational time.

3.1 Linearization of the choice model

We use a discrete choice model to model the demand. The set of all potential alternatives, called the choice set, is denoted by C (i), and the population consists of N customers ($n > 1$). The preference structure of customers is represented with a utility function, which associates a score with each alternative $i \in C_n$ (the set of alternatives considered by customer n):

$$U_{in} = V_{in} + \varepsilon_{in}, \qquad \forall i \in C_n, n, \tag{1}$$

where V_{in} denotes the deterministic part of the utility function, which includes everything that can be modeled by the analyst, and ε_{in} the error term, which captures everything that has not been included explicitly in the model. The behavioral assumption is that customer n chooses alternative i if the corresponding utility is the largest within the choice set C_n.

We assume that V_{in} is linear in the endogenous variables (x^e) of the modeling framework (the ones involved in both the choice and the MILP models). This is not required for the derivation of the choice model, but important in our context for its integration in an MILP formulation. The probabilistic nature of the choice model is addressed with simulation by generating R draws from the distribution of the random term ε_{in}. In this way, the utility associated with alternative $i \in C_n$ by customer n in scenario r is a linear function of the endogenous variables:

$$U_{inr} = \overbrace{\sum_k \beta_k x^e_{ink} + g^d_{in}(x^d_{in})}^{V_{in}} + \xi_{inr}, \qquad \forall i \in C_n, n, r, \tag{2}$$

where x^d denotes the (exogenous) variables that explain the choice and that are not involved in the MILP model, and ξ_{inr} denotes the r-th draw from the distribution of ε_{in}. We model the choice with the binary variables w_{inr}, which take value 1 if alternative i is chosen by individual n in scenario r, and 0 otherwise. The demand of alternative $i \in C$ can therefore be obtained by averaging the sum of the choice variables over R:

$$D_i = \frac{1}{R} \sum_{r=1}^{R} \sum_{n=1}^{N} w_{inr}, \qquad \forall i \in C. \tag{3}$$

3.2 Benefit maximization problem

For the sake of illustration, we embed this linear characterization of a discrete choice model into the demand-based benefit maximization model. We consider an operator that aims at finding the best strategy in terms of pricing and capacity allocation in order to maximize its benefit. We assume that it sells services to a market, each of them at a certain price and with a certain capacity, both to be decided. In a benefit maximization context, we need to account for competition, since otherwise customers are captive, which makes the problem unbounded. To this end, we define an opt-out option to capture the customers leaving the market. We denote it by $i = 0$ and we assume it is available to all customers ($0 \in C_n, \forall n$).

We consider the price as the only endogenous variable in the utility function (2). We define $p_{in} \in \mathbb{R}$ as the price that customer n must pay to access service $i > 0$. The capacity c_i of service $i > 0$ is modeled in a discrete fashion by defining a list of Q feasible values for the capacity: $c_{i1}, \ldots, c_{iQ}$, from which at most one can be chosen (i.e., it is still possible for the operator not to offer a service). Thus, service i is duplicated Q times, each instance being associated with the same utility function, but with a different capacity level. We represent this decision with the binary variables y_{iq}, which take value 1 if service i is offered with capacity c_{iq}, and 0 otherwise.

The objective function calculates the difference between the expected gains obtained from each service $i > 0$ and the associated operating costs. The expected gain obtained from service $i > 0$ is denoted by G_i, and can be derived directly from the demand expression (3) and the price specification. However, this introduces a nonlinearity due to the product of the variables w_{inr} and p_{in}. This product can be easily linearized if an upper bound for the latter is known, which in this case can be set by the operator. We define the variable η_{inr} to capture the product $w_{inr}p_{in}$.

We assume that the operating cost of service $i > 0$ at capacity level c_{iq} is composed of a fixed cost associated with operating the service (f_{iq}) and a variable cost associated with each sold unit of the service (v_{iq}). The resulting objective function is the following:

$$\sum_{i>0} \Bigg[\overbrace{\frac{1}{R} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \eta_{iqnr}}^{G_i} - \overbrace{\sum_{q=1}^{Q} \Big(f_{iq} + v_{iq}c_{iq}\Big) y_{iq}}^{C_i} \Bigg]. \tag{4}$$

Note that the variables η_{iqnr} model the product $w_{iqnr}p_{in}$, and are the natural extension of the variables η_{inr} when accounting for the different capacity levels, i.e., when defining the Q duplicates of each alternative $i > 0$. This distinction is not necessary for the utility variables U_{inr}, as utility remains the same across capacity levels.

Table1 summarizes the main notations used in the model for the reader's convenience, organized by sets, parameters, variables and aggregated quantities. The complete formulation of the demand-based benefit maximization problem is included in Figure 1.

	Name	Description
Sets	C	Set of all potential services (indexed by $i > 0$, $i = 0$ denotes the opt-out option)
	J	Number of services in C
	N	Number of customers in the population (indexed by $n \geq 1$)
	R	Number of draws from the distribution of ε_{in} (indexed by r)
	Q	Number of capacity levels (indexed by q)
Parameters	c_{iq}	Capacity of service i for the q-th level
	ξ_{inr}	Draw from the distribution of ε_{in}
	ℓ_{inr}	Lower bound on U_{inr}
	m_{inr}	Upper bound on U_{inr}
	ℓ_{nr}	Smallest lower bound ℓ_{inr} across services
	m_{nr}	Largest upper bound m_{inr} across services
	M_{inr}	$m_{inr} - \ell_{nr}$
	M_{nr}	$m_{nr} - \ell_{nr}$
	c_{iq}	Capacity level q associated with service i
	a_{in}	Lower bound on p_{in}
	b_{in}	Upper bound on p_{in}
	f_{iq}	Fixed cost associated with service i at capacity level c_{iq}
	v_{iq}	Cost per unit sold of service i at capacity level c_{iq}
Variables	U_{inr}	Utility associated with service i by customer n in scenario r
	y_{iq}	1 if service i is offered with capacity level c_{iq}, 0 otherwise
	y_{iqn}	1 if service i is offered with capacity level c_{iq} to individual n, 0 otherwise
	y_{iqnr}	1 if service i with capacity level c_{iq} is available to individual n in scenario r, 0 otherwise
	z_{iqnr}	U_{inr} if $y_{iqnr} = 1$, and ℓ_{nr} if $y_{iqnr} = 0$
	w_{iqnr}	1 if $(i, q) = \text{argmax}\{U_{nr}\}$, and 0 otherwise
	U_{nr}	$\max_{i,q} z_{iqnr}$
	p_{in}	Price that individual n has to pay to access service i
	η_{iqnr}	$p_{in}w_{iqnr}$
Quant.	D_i	Expected demand of service i
	G_i	Expected gain obtained from service $i > 0$
	C_i	Total cost associated with service $i > 0$

Table 1: Main notations used in the demand-based maximization problem

$$
\begin{array}{llrr}
\max & \displaystyle\sum_{i>0}\left[\overbrace{\frac{1}{R}\sum_{q=1}^{Q}\sum_{n=1}^{N}\sum_{r=1}^{R}\eta_{iqnr}}^{G_i}-\overbrace{\sum_{q=1}^{Q}\Big(f_{iq}+v_{iq}c_{iq}\Big)y_{iq}}^{C_i}\right] & & (4)\\
\text{subject to} & U_{inr}=\beta_{in}p_{in}+g^d_{in}(x^d_{in})+\xi_{inr} & \forall i\in C_n,n,r & (2)\\
& y_{iqn}\leq y_{iq} & \forall i>0,q,n & (5)\\
& y_{iqn}=0 & \forall i\notin C_n,q,n & (6)\\
& y_{iqnr}\leq y_{iqn} & \forall i>0,q,n,r & (7)\\
& \displaystyle\sum_{q=1}^{Q}y_{iq}\leq 1 & \forall i>0 & (8)\\
& \ell_{nr}\leq z_{iqnr} & \forall i,q,n,r & (9)\\
& z_{iqnr}\leq \ell_{nr}+M_{inr}y_{iqnr} & \forall i,q,n,r & (10)\\
& U_{inr}-M_{inr}(1-y_{iqnr})\leq z_{iqnr} & \forall i,q,n,r & (11)\\
& z_{iqnr}\leq U_{inr} & \forall i,q,n,r & (12)\\
& z_{iqnr}\leq U_{nr} & \forall i,q,n,r & (13)\\
& U_{nr}\leq z_{iqnr}+M_{nr}(1-w_{iqnr}) & \forall i,q,n,r & (14)\\
& \displaystyle\sum_{i\in C}\sum_{q=1}^{Q}w_{iqnr}=1 & \forall n,r & (15)\\
& w_{iqnr}\leq y_{iqnr} & \forall i>0,q,n,r & (16)\\
& \displaystyle\sum_{m=1}^{n-1}w_{iqmr}\leq (c_{iq}-1)y_{iqnr}+(n-1)(1-y_{iqnr}) & \forall i>0,q,n>c_{iq},r & (17)\\
& c_{iq}(y_{iqn}-y_{iqnr})\leq \displaystyle\sum_{m=1}^{n-1}w_{iqmr} & \forall i>0,q,n,r & (18)\\
& a_{in}w_{iqnr}\leq \eta_{iqnr} & \forall i>0,q,n,r & (19)\\
& \eta_{iqnr}\leq b_{in}w_{iqnr} & \forall i>0,q,n,r & (20)\\
& p_{in}-(1-w_{iqnr})b_{in}\leq \eta_{iqnr} & \forall i>0,q,n,r & (21)\\
& \eta_{iqnr}\leq p_{in}-(1-w_{iqnr})a_{in} & \forall i>0,q,n,r & (22)
\end{array}
$$

Figure 1: Demand-based benefit maximization problem

Constraints (5)–(8) have to do with the availability of the services. The variables y_{iqn} model the availability at operator level, and take value 1 if the operator decides to offer service $i>0$ to customer n at capacity level c_{iq}. The variables y_{iqnr} model the availability at scenario level, and take value 0 when the capacity has been reached. We assume an exogenously given priority list of customers to decide who has access to a certain service when its demand is larger than its capacity. This list simply determines the order in which customers are processed.

In order to avoid that an unavailable service is chosen, we introduce the concept of discounted utility (z_{iqnr}), which is the utility itself if the service is available, and a low value otherwise. Constraints (9)–(12) provide a linear representation of this definition, and constraint (16) sets the choice variables to 0 in case of unavailability. We also define the variable U_{nr} to capture the highest discounted utility across alternatives and capacity levels for each customer n and scenario r. This maximum is linearly characterized with constraints (13)–(14). Constraint (15) imposes that only one service can be chosen by customer n in scenario r.

Constraints (17)–(18) deal with the capacity allocation. More precisely, the former is binding when the capacity has been reached, and set the variables y_{iqnr} to 0 so that the service is not accessible, whereas the latter is active when the capacity has not been reached by setting the variables y_{iqnr} to 1. Finally, constraints (19)–(22) linearize the product $\eta_{iqnr} = w_{iqnr} p_{in}$.

3.3 Computational results

For the proof-of-concept, we consider the case study of a parking services operator (Ibeas *et al.*, 2014), where the authors characterize a mixture of logit models to describe the behavior of potential car park users when choosing a parking place. For the sake of illustration, we consider a sample of $N = 50$ customers and $R = 50$ draws. The choice set consists of three services: paid on-street parking (PSP), paid parking in an underground car park (PUP) and free on-street parking (FSP), which is considered as the opt-out option (it does not provide any revenue).

In one of the experiments performed in Pacheco *et al.* (2017), we test two different approaches concerning the potential services: (1) the operator can decide if the services are offered or not, and (2) it is forced to offer all services. In both cases, the price and the capacity of PSP and PUP are to be decided by the model. Table 2 shows the complexity of these models with respect to the solution time: almost 19 hours for approach 1 and almost 34 for approach 2. The difference in time between approaches comes from the different flexibility being assumed. In terms of benefit, the first approach is more profitable, as expected, because the operator can freely decide on its resources.

	Solution	**Capacity**		**Demand**			**Prices**		
Approach	time (h)	PSP	PUP	PSP	PUP	FSP	PSP	PUP	Benefit
(1)	18.7	20	-	19.4	-	30.6	0.76	-	6.27
(2)	33.7	15	5	14.8	4.56	30.7	0.76	1.32	4.99

Table 2: Price and capacity allocation for approaches 1 and 2

4 Decomposition by customers and draws

As introduced in Section 1, there are two dimensions along which the problem can be decomposed: the customers and the draws. The formulation in Figure 1, however, cannot be directly decomposed in independent subproblems for each customer n and scenario r, as all the customers and draws are combined together in the objective function, and the customers are also coupled in the capacity constraints.

In this section, we include the decomposition of a simplified version of the demand-based benefit maximization problem to illustrate the decomposition by customers and draws. More precisely, we assume unlimited capacity for all services, i.e., there is room for all customers in all services; and we set the operating costs to 0, which converts the problem into a revenue maximization problem. We define this decomposition strategy as a first step to decompose more complex versions of the problem: including capacity, operating costs, etc. We are currently investigating the demand-based revenue maximization problem for the capacitated case.

Formulation In the uncapacitated case, $c_{iq} = \infty, \forall i > 0, q$, which enables us to get rid of the following elements: the index q; the variables y_{inr} and z_{inr}, as they are related to the availability of the services due to capacity restrictions; and constraints (7), (9)–(12), and (16)–(18). Constraints (13)–(14) can be directly written in terms of the utility variables U_{inr}. Moreover, the zero cost assumption allows us to dispense with the variables y_{iq} and y_{iqn}. Indeed, the former are not needed as the capacity is given, and the latter can be ignored as the services are offered to all customers because we do not consider the associated costs.

The resulting formulation can be found in Figure 2. The objective function (25) is the sum of the revenues obtained from all services, constraint (2) represents the utility function, constraint (15) ensures that only one service is chosen per customer and draw, constraints (19)–(22) are the linearizing constraints of the product $\eta_{inr} = p_{in} w_{inr}$, and constraints (26)–(27) characterize the highest utility by means of the utility variables U_{inr}.

The variables of this problem are: η_{inr}, U_{inr}, p_{in}, U_{nr} and w_{inr}. We notice that p_{in} does not let us to decompose the problem into independent subproblems for each customer n and scenario r, since the price is the same across draws. In order to overcome this limitation, we create R copies of this variable, we denote them by p_{inr}, and we add the following constraints:

$$p_{inr} = p_{inR}, \qquad \forall i > 0, n, r = 1, \tag{23}$$

$$p_{inr} = p_{in(r-1)}, \qquad \forall i > 0, n, r > 1. \tag{24}$$

We replace p_{in} by p_{inr} in constraints (2) and (21)–(22), and we relax constraints (23)–(24) by transferring them to the objective function with associated Lagrangian multipliers $\lambda_{inr} \in \mathbb{R}, \forall i > 0, n, r$. The resulting problem is included in Figure 3. In order to provide a compact expression for the objective function, p_{in0} (obtained when $r = 1$) represents p_{inR}.

$$
\begin{array}{llrr}
\max & \sum_{i>0} \frac{1}{R} \sum_{n=1}^{N} \sum_{r=1}^{R} \eta_{inr} & & (25) \\
\text{subject to} & U_{inr} = \beta_{in} p_{in} + g_{in}^{d}(x_{in}^{d}) + \xi_{inr} & \forall i \in C_n, n, r & (2) \\
& \sum_{i \in C} w_{inr} = 1 & \forall n, r & (15) \\
& a_{in} w_{inr} \leq \eta_{inr} & \forall i > 0, n, r & (19) \\
& \eta_{inr} \leq b_{in} w_{inr} & \forall i > 0, n, r & (20) \\
& p_{in} - (1 - w_{inr}) b_{in} \leq \eta_{inr} & \forall i > 0, n, r & (21) \\
& \eta_{inr} \leq p_{in} - (1 - w_{inr}) a_{in} & \forall i > 0, n, r & (22) \\
& U_{inr} \leq U_{nr} & \forall i, n, r & (26) \\
& U_{nr} \leq U_{inr} + M_{nr}(1 - w_{inr}) & \forall i, n, r & (27)
\end{array}
$$

Figure 2: Demand-based revenue maximization problem (uncapacitated case)

$$
\begin{array}{llrr}
\max & \sum_{i>0} \frac{1}{R} \sum_{n=1}^{N} \sum_{r=1}^{R} \eta_{inr} + \sum_{i>0} \sum_{n=1}^{N} \sum_{r=1}^{R} \lambda_{inr}(p_{inr} - p_{in(r-1)}) & & (28) \\
\text{subject to} & U_{inr} = \beta_{in} p_{inr} + g_{in}^{d}(x_{in}^{d}) + \xi_{inr} & \forall i \in C_n, n, r & (2) \\
& U_{inr} \leq U_{nr} & \forall i, n, r & (26) \\
& U_{nr} \leq U_{inr} + M_{nr}(1 - w_{inr}) & \forall i, n, r & (27) \\
& \sum_{i \in C} w_{inr} = 1 & \forall n, r & (15) \\
& a_{in} w_{inr} \leq \eta_{inr} & \forall i > 0, n, r & (19) \\
& \eta_{inr} \leq b_{in} w_{inr} & \forall i > 0, n, r & (20) \\
& p_{inr} - (1 - w_{inr}) b_{in} \leq \eta_{inr} & \forall i > 0, n, r & (21) \\
& \eta_{inr} \leq p_{inr} - (1 - w_{inr}) a_{in} & \forall i > 0, n, r & (22)
\end{array}
$$

Figure 3: Lagrangian relaxation for the demand-based revenue maximization problem (uncapacitated case)

We can now characterize a subproblem for each customer n and draw r. The objective function of each subproblem consists of the corresponding η_{inr} variable (weighted by $1/R$) and the terms depending on the associated p_{inr} variables. Figure 4 shows the resulting subproblem for a given n and r, where $\lambda_{in(R+1)}$ (obtained when $r = R$) represents λ_{in1}.

$$
\begin{array}{llrr}
\max & \sum_{i>0} \left[\frac{1}{R}\eta_{inr} + (\lambda_{inr} - \lambda_{in(r+1)})p_{inr} \right] & & (29) \\
\text{subject to} & U_{inr} = \beta_{in}p_{inr} + g_{in}^{d}(x_{in}^{d}) + \xi_{inr} & \forall i \in C_n & (30) \\
& U_{inr} \leq U_{nr} & \forall i & (31) \\
& U_{nr} \leq U_{inr} + M_{nr}(1 - w_{inr}) & \forall i & (32) \\
& \sum_{i \in C} w_{inr} = 1 & & (33) \\
& a_{in}w_{inr} \leq \eta_{inr} & \forall i > 0 & (34) \\
& \eta_{inr} \leq b_{in}w_{inr} & \forall i > 0 & (35) \\
& p_{inr} - (1 - w_{inr})b_{in} \leq \eta_{inr} & \forall i > 0 & (36) \\
& \eta_{inr} \leq p_{inr} - (1 - w_{inr})a_{in} & \forall i > 0 & (37)
\end{array}
$$

Figure 4: Subproblem associated with customer n and scenario r for the demand-based revenue maximization problem for the uncapacitated case

Algorithm For given values of the multipliers λ_{inr}, the subproblem in Figure 4 can be directly solved with a commercial software like Gurobi or CPLEX. In this case, we address the combinatorial nature of the problem with enumeration to come up with linear problems (LP). Briefly, for each customer and draw, we iterate over the services in C_n, and at each iteration we assume that service i is chosen. Under this assumption, the problem (38)–(41) needs to be solved.

$$
\begin{array}{llrr}
\max & \left(\frac{1}{R} + \lambda_{inr} - \lambda_{in(r+1)}\right)p_{inr} + \sum_{j \in C_n, j \neq i} (\lambda_{jnr} - \lambda_{jn(r+1)})p_{jnr} & & (38) \\
\text{subject to} & U_{inr} \geq U_{jnr} & \forall j \in C_n, j \neq i & (39) \\
& p_{jnr} \geq a_{jn} & \forall j \in C_n & (40) \\
& p_{jnr} \leq b_{jn} & \forall j \in C_n & (41)
\end{array}
$$

We notice that such a problem can be infeasible if it does not exist any price between the bounds that makes service i to be the one with the highest utility. Nevertheless, even if the problem (38)–(41) is infeasible $\forall i > 0$, the LP associated with the opt-out option is always feasible. Therefore, service i will be chosen by customer n in scenario r if its objective function is the highest among the feasible LPs, and its price, as well as the prices for the unchosen services, are determined when solving the LP associated with this service.

Subgradient method As mentioned in Section 2, an iterative method is used to update the values of the Lagrangian multipliers, i.e., to solve the Lagrangian dual. The main steps of the subgradient method are the following:

1. **Initialization:** set a number of iterations K, initialize $k = 0$ and choose starting values for the Lagrangian multipliers. $\lambda^0_{inr} = 0, \forall i > 0, n, r$
2. **Subgradients:** obtain subgradients $s^k_{inr} = p^k_{inr} - p^k_{in(r-1)}, \forall i > 0, n, r$, of the objective function (28) at λ^k_{inr}. If $s^k_{inr} = 0, \forall i > 0, n, r$, then **STOP** (the optimal value has been reached).
3. **Update:** compute the Lagrangian multipliers for the following iteration:

$$\lambda^{k+1}_{inr} = \lambda^k_{inr} + \gamma^k s^k_{inr}, \qquad \forall i > 0, n, r. \tag{42}$$

4. **Stopping criteria:** increment k; if $k = K$, then **STOP**, otherwise go to step 2.

The step size is denoted by γ^k and enables to follow the subgradient at the current position in order to reach points with a lower function value. Its definition is of crucial importance because the speed of convergence depends strongly on it. There exist several approaches in the literature to define the step size. In the following, we consider the simple step size

$$\gamma^k = \frac{1}{k+1}, \qquad \forall k, \tag{43}$$

and the formula proposed by Held *et al.* (1974) for adapting the step size

$$\gamma^k = \mu^k \frac{Z^* - Z(\lambda^k)}{\sum_{i>0} \sum_{n=1}^{N} \sum_{r=1}^{R} (s^k_{inr})^2}, \qquad \forall k, \tag{44}$$

where Z^* is the value of the best solution for the original problem found so far, $Z(\lambda^k)$ is the current value of the objective function of the Lagrangian dual, and μ^k is a decreasing adaption parameter with $0 \leq \mu^0 \leq 2$. In our case, we start with $\mu^0 = 1$ and we reduce it by a factor of 2 whenever $Z(\lambda^k)$ has failed to decrease in the last 2 iterations. For the sake of illustration, we use the optimal value of the objective function of the exact method as Z^*, but it can initially be set to 0 and then updated using the solutions that are obtained in those iterations in which the Lagrangian problem solution turns out to be feasible in the original problem.

Preliminary results We use the case study defined in Section 3.3 to perform some preliminary experiments. We first solve the exact method (Figure 2), and we then consider the Lagrangian relaxation approach with the two different step sizes mentioned above. For each iteration of the Lagrangian decomposition scheme, we iterate over the customers and the draws, and for each customer n and scenario r, we solve the corresponding subproblem with the algorithm described previously. The Lagrangian multipliers are updated with the subgradient method, and the procedure finishes as soon as any of the two above mentioned stopping criteria is satisfied.

Table 3 shows the computational time and the obtained revenues for the exact method (solved with CPLEX) and for the Lagrangian decomposition with $K = 50$ and both step sizes. All the experiments were performed with a 2.5 GHz Intel Core i7 processor. For a small number of draws ($R = 25$), the exact method is faster than the Lagrangian decomposition technique, but for larger values of R, the solution time is remarkably lower. Furthermore, the growing in computational time with the Lagrangian relaxation method is controlled, in the sense that each iteration of the method has a similar cost. At first glance, the values of the revenue are in line to those obtained with the exact method, but a proper assessment is needed. We highlight the fact that a high number of draws, that previously with the exact method could not be considered due to computational limitations, can be considered now.

	Solution time (s)			**Revenue**		
R	Exact	(43)	(44)	Exact	(43)	(44)
25	39.65	46.93	47.38	28.68	27.50	28.77
50	217.10	94.54	93.05	28.26	28.56	29.26
75	368.58	139.58	142.18	27.64	28.09	28.85

Table 3: Solution times and revenue for the demand-based revenue maximization problem for the uncapacitated case

In terms of the expected demand, we observe some discrepancies in Table 4. More precisely, for both step sizes, the demand of PSP is underestimated, whereas the demand of the opt-out option is clearly higher than the one obtained with the exact method. Between step sizes, the one defined in (44) provides closer values. We also include the share of subgradients that are equal to 0, which indicates the percentage of duplicates of the price variables that are the same among them. We observe that the shares are low, but we need to take into account that the price is a continuous variable, and equality is hard to be satisfied. For future experiments, we should define a threshold under which two price variables are considered the same.

	Demand PSP			**Demand PUP**			**Demand FSP**			**Zero subgradients (%)**	
R	Exact	(43)	(44)	Exact	(43)	(44)	Exact	(43)	(44)	(43)	(44)
25	18.4	13.8	15.3	17.2	16.0	15.8	14.4	20.1	18.8	21.0	21.3
50	20.5	13.8	15.2	16.0	17.1	16.6	13.5	19.1	18.2	21.9	21.8
75	18.8	13.5	14.7	16.6	16.9	16.7	14.6	19.6	18.7	22.1	22.5

Table 4: Expected demand for each service and share of zero subgradients for the demand-based revenue maximization problem for the uncapacitated case

5 Decomposition by operator and customers

The technique described in Section 4 illustrates the procedure to decompose the demand-based revenue maximization problem for the uncapacitated case by customer and draw. The inclusion of capacity restrictions and/or operating costs associated with the services will make the problem much more difficult, and the decompositon by both customer and draw might fail. As mentioned above, we are currently exploring additional decomposition strategies for such problems.

The ultimate goal is to use these strategies within the general Lagrangian relaxation method for the demand-based benefit maximization problem. As mentioned in Section 1, this problem contains decisions concerning two different agents: the operator and the customers. The former wants to decide on the price and the capacity of the potential offered services, whereas the latter are willing to pay a certain price to access their preferred service, in the sense that if this price is not appropriate, they might leave the market.

We can therefore define a Lagrangian decomposition scheme composed of two subproblems: one concerning the decisions of the operator (the operator subproblem) and one the decisions of the customers (the customer subproblem). In order to separate the formulation in Figure 1 into these subproblems, we identify the constraints associated with each of them:

- Operator subproblem: (5), (8), (19)–(22), and
- Customer subproblem: (2), (6)–(7), (9)–(18), (19)–(22).

We note that constraints (19)–(22) belong to both subproblems because price is the endogenous variable of this formulation. The following variables are common in both subproblems: y_{iqn}, y_{iqnr}, w_{iqnr}, p_{in} and η_{iqnr}. Since duplicating all the variables to define two separable subproblems will complicate the Lagrangian relaxation approach, we consider the following procedure:

1. Duplicate the choice variables: $w'_{iqnr} = w_{iqnr}, \forall i, q, n, r$. We note that this constraint is equivalent to constraints (45)–(46):

$$w_{iqnr} \leq w'_{iqnr}, \qquad \forall i, q, n, r, \tag{45}$$

$$\sum_{i \in C} \sum_{q=1}^{Q} w'_{iqnr} = 1, \qquad \forall n, r. \tag{46}$$

   The advantage of constraints (45)–(46) is twofold:
   a) the replacement of the equality by the inequality in (45) introduces non-negative Lagrangian multipliers (instead of unconstrained ones), and
   b) the introduction of redundant assignment constraints (46) strengths the Lagrangian

subproblem.

2. Replace constraint (5) (which contains the variables y_{iqn}, that are common in both subproblems, and y_{iq}, that are only in the operator subproblem) by the following constraint linking the duplicate of the choice variables w'_{iqnr} and the y_{iq} variables:

$$w'_{iqnr} = w_{iqnr} \overset{(16)}{\leq} y_{iqnr} \overset{(7)}{\leq} y_{iqn} \overset{(5)}{\leq} y_{iq}, \qquad \forall i > 0, q, n, r. \tag{47}$$

3. Include constraints (19)–(22) only in the customer subproblem.

With this procedure, we can separate the demand-based benefit maximization problem into two subproblems that depend on disjointed sets of variables:

- Operator subproblem: variables y_{iq} and w'_{iqnr}, and constraints (8), (46)–(47), and
- Customer subproblem: variables y_{iqn}, y_{iqnr}, U_{inr}, z_{iqnr}, U_{nr}, w_{iqnr}, p_{in} and η_{iqnr}, and constraints (2), (6)–(7), (9)–(22).

The objective function associated with each subproblem is obtained by transferring constraint (45) with non-negative Lagrangian multipliers $\theta_{iqnr}, \forall i, q, n, r$:

$$\begin{aligned} Z(\theta) &= \sum_{i>0} G_i - C_i + \sum_{i \in C} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr}(w'_{iqnr} - w_{iqnr}) = \\ &= \overbrace{\sum_{i>0} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \frac{1}{R}\eta_{iqnr} - \sum_{i \in C} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr} w_{iqnr}}^{Z_c(\theta)} \\ &\quad \overbrace{- \sum_{i>0} \sum_{q=1}^{Q} (f_{iq} + v_{iq}c_{iq}) y_{iq} + \sum_{i \in C} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr} w'_{iqnr}}^{Z_o(\theta)}, \end{aligned} \tag{48}$$

where the term denoted by $Z_o(\theta)$ corresponds to the operator subproblem and the term $Z_c(\theta)$ to the customer subproblem.

Operator subproblem We note that with the current formulation of the operator subproblem, w'_{iqnr} is not set to 0 when $i \notin C_n$. Thus, we need to add the following constraint to set the duplicates of the choice variable to 0 when the service is not considered by the customer:

$$w'_{iqnr} = 0, \qquad \forall i \notin C_n, q, n, r. \tag{49}$$

Furthermore, we can add the following valid inequality to the model to ensure that the capacity of facilities is not exceeded:

$$\sum_{n=1}^{N} w'_{iqnr} \leq c_{iq} y_{iq}, \qquad \forall i > 0, q, r. \tag{50}$$

This enables us to express the operator subproblem as an extension to the classical Capacitated Facility Location Problem (CFLP), where:

- C represents the set for the potential locations for the facilities,
- N represents the index for the set of users,
- y_{iq} represent the location decision variables, and
- w'_{iqnr} represent the allocation decision variables.

The operator subproblem written as a CFLP is included in Figure 5.

$$\begin{array}{llll}
\max & Z_o(\theta) = \sum_{i \in C} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr} w'_{iqnr} - \sum_{i>0} \sum_{q=1}^{Q} (f_{iq} + v_{iq} c_{iq}) y_{iq} & & (51) \\
\text{subject to} & \sum_{q=1}^{Q} y_{iq} \leq 1 & \forall i > 0 & (7) \\
& \sum_{i \in C} \sum_{q=1}^{Q} w'_{iqnr} = 1 & \forall n, r & (46) \\
& w'_{iqnr} \leq y_{iq} & \forall i > 0, q, n, r & (47) \\
& w'_{iqnr} = 0 & \forall i \notin C_n, q, n, r & (49) \\
& \sum_{n=1}^{N} w'_{iqnr} \leq c_{iq} y_{iq} & \forall i > 0, q, r & (50)
\end{array}$$

Figure 5: Operator subproblem for the demand-based benefit maximization problem

The objective function can be interpreted as the maximization of the net profit by the operator, which is defined as the difference between the revenuem generated from the serviced customers and the cost of the location of the selected facilities. Constraints (46) are equivalent to the ones in the CFLP guaranteeing that each customer is served from one facility. Constraints (50) play a double role in the CFLP: they ensure that the capacity of facilities is not exceeded and they prevent users from being allocated to non-open facilities. Constraints (7) and (49) are necessary for our context but could also be part of a CFLP with equivalent characteristics.

Even if the CFLP is NP-hard, solving directly the MILP might work for small instances, and there exist several works in the literature to solve it. One example is Lagrangian relaxation, which in this case consists on relaxing the assignment constraints (46).

Customer subproblem The subproblem dealing with the choices of the customers is defined in Figure 6, and it contains all the remaining information. A decomposition strategy in a similar fashion than the one defined in Section 4 can be characterized to tackle this problem.

$$
\begin{array}{llr}
\max & Z_c(\theta) = \sum_{i>0} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \frac{1}{R} \eta_{iqnr} - \sum_{i \in C} \sum_{q=1}^{Q} \sum_{n=1}^{N} \sum_{r=1}^{R} \theta_{iqnr} w_{iqnr} & (52) \\
\text{subject to} & y_{iqn} = 0 \qquad \forall i \notin C_n, q, n & (6) \\
& y_{iqnr} \leq y_{iqn} \qquad \forall i > 0, q, n, r & (7) \\
& U_{inr} = \beta_{in} p_{in} + g^d_{in}(x^d_{in}) + \xi_{inr} \qquad \forall i, n, r & (2) \\
& \ell_{nr} \leq z_{iqnr} \qquad \forall i, q, n, r & (9) \\
& z_{iqnr} \leq \ell_{nr} + M_{inr} y_{iqnr} \qquad \forall i, q, n, r & (10) \\
& U_{inr} - M_{inr}(1 - y_{iqnr}) \leq z_{iqnr} \qquad \forall i, q, n, r & (11) \\
& z_{iqnr} \leq U_{inr} \qquad \forall i, q, n, r & (12) \\
& z_{iqnr} \leq U_{nr} \qquad \forall i, q, n, r & (13) \\
& U_{nr} \leq z_{iqnr} + M_{nr}(1 - w_{iqnr}) \qquad \forall i, q, n, r & (14) \\
& \sum_{i \in C} \sum_{q=1}^{Q} w_{iqnr} = 1 \qquad \forall n, r & (15) \\
& w_{iqnr} \leq y_{iqnr} \qquad \forall i > 0, q, n, r & (16) \\
& \sum_{m=1}^{n-1} w_{iqmr} \leq (c_{iq} - 1) y_{iqnr} + (n-1)(1 - y_{iqnr}) \qquad \forall i > 0, q, n > c_{iq}, r & (17) \\
& c_{iq}(y_{iqn} - y_{iqnr}) \leq \sum_{m=1}^{n-1} w_{iqmr} \qquad \forall i > 0, q, n > 1, r & (18) \\
& a_{in} w_{iqnr} \leq \eta_{iqnr} \qquad \forall i > 0, q, n, r & (19) \\
& \eta_{iqnr} \leq b_{in} w_{iqnr} \qquad \forall i > 0, q, n, r & (20) \\
& p_{in} - (1 - w_{iqnr}) b_{in} \leq \eta_{iqnr} \qquad \forall i > 0, q, n, r & (21) \\
& \eta_{iqnr} \leq p_{in} - (1 - w_{iqnr}) a_{in} \qquad \forall i > 0, q, n, r & (22)
\end{array}
$$

Figure 6: Customer subproblem for the demand-based benefit maximization problem

6 Conclusions and future work

As mentioned in Section 4, we are currently running additional experiments to assess the validity of the Lagrangian relaxation approach, and we are working on the characterization of other decomposition strategies for more complex versions of the demand-based revenue maximization problem for the uncapacitated case, such as those including a finite capacity and/or the associated operating costs. The final idea is to integrate such strategies with the decomposition scheme by agent-based decisions described in Section 5, which is the procedure addressing the demand-based benefit maximization problem.

Once the decomposition scheme is fully characterized, we are planning to test the formulation in large instances (with a high number of individuals and a high number of draws to be as accurate as possible) where the exact method would fail. Moreover, in order to obtain closer results, we need to refine the subgradient method by increasing the number of iterations and by considering more appropriate step size calculations.

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