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Stefano Bortolomiol

Virginie Lurkin

Michel Bierlaire

Transport and Mobility Laboratory, EPFL

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Stefano Bortolomiol, Michel Bierlaire
Transport and Mobility Laboratory
École Polytechnique Fédérale de Lausanne
EPFL ENAC IIC - TRANSP-OR, Station 18, 1015 Lausanne, Switzerland
stefano.bortolomiol@epfl.ch ,michel.bierlaire@epfl.ch

Virginie Lurkin
Department of Industrial Engineering and Innovation Sciences
Eindhoven University of Technology
Atlas 4.402, De Zaale, Eindhoven, Netherlands
v.j.c.lurkin@tue.nl

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Abstract

Oligopolistic competition occurs when a small number of operators compete for the same pool of customers. This is often the case in transportation, due to reasons such as external regulations, economies of scale and limited capacity of the infrastructure. We present a demand-based optimization approach to study market equilibria in oligopolies. The framework takes into account interactions between demand and supply as well as competition among suppliers. In particular, the preferences of the customers are modelled at a disaggregate level according to random utility theory, while competition is modelled as a multi-leader-follower game. To find equilibrium solutions, we propose a fixed-point optimization model which can incorporate both nonlinear and linearized customer choices probabilities. Due to its complexity, the model can only tackle small-size instances with restricted strategy sets. Finally, we include a preliminary description of a heuristic approach that can be used to efficiently select for all competitors a subset of strategies that have the potential to produce equilibrium or near-equilibrium solutions.

Keywords

competition, oligopolies, equilibrium, demand-based optimization, mixed integer programming

1 Introduction

Competition in transportation is a multifaceted phenomenon. On the demand side, users compete with other users on the road, to avoid congestion and minimize their travel time, and on public transport, to purchase the cheapest tickets or maximize the comfort of their journey. Users also compete with transport operators, which represent the supply side and aim at achieving an optimal configuration of their system and maximize their profit, social welfare or any other relevant objective. Moreover, in non-monopolistic markets, transport operators compete on price, quantity, quality and other features to attract customers, while acting themselves as customers of other service providers like the infrastructure owners, who for example allocate departure and arrival slots at airports or track possessions in railways. These examples show how complex competitive markets are and how daunting of a task it would be to consider all direct interactions between all market agents. The focus of our research is directed to oligopolistic markets, in which we model operator-customers and operator-operator interactions, which fit into the frameworks of Stackelberg games and Nash non-cooperative games respectively. Historically, the airline industry has been the most studied case of oligopoly in transportation, but intercity train and bus operators as well as urban on-demand transport service providers also operate in oligopolistic conditions in an increasing number of cases. The objective of this work is to analyze oligopolistic markets using a demand-based optimization framework that is mathematically sound and computationally tractable, and that could therefore be used to analyze case studies in transportation as well as in other markets.

The rest of the paper is organized as follows. Section 2 contains the literature review. Section 3 presents the modelling framework, which includes demand, supply and market interactions, and details a fixed-point optimization model to find market equilibria. Section 4 reports the results of some numerical experiments which identify strengths and weaknesses of the proposed formulations. Finally, Section 5 summarizes the outcomes of this work and describes the next steps of the research project, which include the development of a heuristic approach to generate restricted sets of strategies to be used as input to the fixed-point optimization model.

2 Literature review

In this section we start by introducing the fundamental game theoretical concepts and results that are relevant to study equilibrium problems. Then, we illustrate the main approaches that have been proposed to study oligopolistic markets and we explain why alternative methods are needed when modelling demand at a disaggregate level. Finally, we motivate the need for the integration of demand and supply into a unique framework and we look into the literature on demand-based optimization models.

2.1 Equilibrium

The discipline that studies competition between groups of decision-makers when individual choices jointly determine the outcome is known as game theory. An overview of the principal game theory concepts can be found in Osborne and Rubinstein (1994). We consider here two types of non-cooperative games, namely the Nash game and the Stackelberg game. The Nash game (Nash, 1951) considers players who have equal status and who can affect their competitors' decisions by changing their strategy unilaterally. In such setting, we define as Nash equilibrium solution of the game a state in which no player can improve its payoff by unilaterally changing his decision. The Stackelberg game (Von Stackelberg, 1934) features two players, identified as leader and follower, both trying to optimize their own objective function. The leader is assumed to know the follower's best responses to all the leader's strategies, and will therefore optimize its decisions accordingly.

A number of definitions should be introduced before discussing equilibria of non-cooperative games. A game is finite when all players have a finite number of moves and a finite number of strategies at each move, otherwise it is infinite. In terms of strategies, in a pure strategy game all players choose one move from their strategy set, while in a mixed strategy game they assign a probability to each pure strategy. Finally, players can have a continuous or a non-continuous payoff function, with the former option being impossible for finite pure strategy games.

Two categories of problems dealing with equilibria are recurrent in the literature: (i) proving the existence of a Nash equilibrium for a given game, and (ii) finding one Nash equilibrium solution of a game. For the first category, Nash (1950) demonstrates that finite games have at least one mixed strategy equilibrium solution, while Glicksberg (1952) extends Nash's results to games with compact strategy sets and continuous payoff functions. For the second category, it has been demonstrated that the problem of finding Nash equilibria belongs to the complexity class PPAD (Papadimitriou, 1994). Well-known algorithms to find Nash equilibria include the Lemke-Howson algorithm (Lemke and Howson, 1964) and the Porter-Nudelman-Shoham algorithm (Porter *et al.*, 2008), while a mixed integer program formulation to find Nash equilibria is introduced in Sandholm *et al.* (2005). An overview of several algorithmic methods used to find Nash equilibria under different circumstances is provided by Nisan *et al.* (2007).

While the most important theoretical concept used to analyze both perfectly competitive and oligopolistic markets is that of equilibrium, in the economic literature it is widely acknowledged that the utility maximization behavior is unlikely to hold true for all agents. The situation in which no nonmaximizing agent would gain a significant amount by becoming a maximizer is defined as near-rational equilibrium or epsilon-equilibrium. Akerlof *et al.* (1985) consider an economy in which a fraction of the population does not maximize its utility. Inertia and the use of rules of thumb in the decision-making process are mentioned as reasons that explain non-rational behavior. It is shown that first-order errors made by an agent on its decision variables cause profit losses that are only second-order small, but in turn the changes in the equilibrium can be first-order. The implication is that the equilibrium solutions derived from models with strictly maximizing behaviors are not robust and small deviations from rationality can make significant differences in equilibria.

2.2 Competition in oligopolies

The theory of competition in non-cooperative games has been extensively used to analyze oligopolistic markets, where a small number of firms are active and have non-negligible market power. The first contributions date back to the seminal works by Cournot (1838) and Bertrand (1883), which analyze a market where an homogeneous product is sold to an homogeneous population and where firms compete on quantity and price, respectively. Hotelling (1929) proposes a duopolistic game in which firms decide on the location of production and on the price of the product, while the homogeneous and inelastic demand is distributed along a line, which affects the cost of transportation from producers to customers. By using a spatial model, he questions the assumption used by Bertrand (1883) and Edgeworth (1925) that consumers abruptly change their product choice when a seller marginally decreases its price. Hotelling argues that stability in such competitive game is achieved when there is little product differentiation across producers and identifies price discrimination and elastic demand as two modification elements needed to make the model of the market more realistic. Gabszewicz and Thisse (1979) consider consumers having identical preferences but variable incomes who make indivisible and mutually exclusive purchases. In a duopolistic market, income differentiation is shown to support product differentiation, contrarily to what Hotelling's model seemed to suggest. The authors also notice that (i) the existence of a Cournot equilibrium requires the continuity in the demand function and (ii) the proof of existence based on fixed-point arguments is based on the quasi-concavity of the profit functions. Murphy *et al.* (1982) propose a mathematical programming approach to find market equilibria in an oligopolistic market supplying an homogeneous product, in which firms must determine their production levels. Assumptions are made on the revenue curves which must be concave, on the demand curve which must be continuously differentiable and on the supply curve which must be convex and continuously differentiable. When these assumptions hold, the equilibrium solution can be found by solving the Karush-Kuhn-Tucker conditions for the optimization problems of the firms. Maskin and Tirole (1988) propose a class of alternating-move models of duopoly in which firms are committed to a certain action in the short-term, thus allowing time for the other firm to react. Firms can choose actions from a bounded set and profits are exclusively dependant on the current actions. The latter assumption is in contrast with the repeated game treatment of oligopolies, proposed for example by Rotemberg and Saloner (1986). Due to the time-independence assumption, equilibrium solutions in the resulting game are Markov perfect equilibria. The general model is applied to a market where firms compete on quantities and where large fixed costs are present which make the market a natural monopoly.

The researches presented above, together with more recent contributions, share the common finding that the existence of an equilibrium is only guaranteed thanks to assumptions made on the demand side which are often simplistic and unsuitable to model real-life markets. In particular, in our work we are interested in discrete goods markets in which individuals have different tastes and socio-economic characteristics which influence their decision when making unitary and mutually exclusive purchases. This situation is frequent in the transportation sector. By modelling demand at a disaggregate level using discrete choice models, the profit function of the firms becomes non-concave, thus invalidating any results about the existence and the uniqueness of equilibrium solutions for such markets. For this reason, in our work we want to explore alternative mathematical models and algorithms to study oligopolistic markets, while retaining the microeconomic foundations on which the presented literature is grounded.

2.3 Demand-based optimization

Demand-based (or choice-based) optimization models have been proposed to analyze customer behavior at a disaggregate level and incorporate it into the optimization problem of the suppliers. Thanks to a better estimation of customer preferences, suppliers can then improve many of their strategic decisions.

Historically, demand modelling and supply optimization have been treated as two separate problems, requiring different methodologies to be solved. On the demand side, discrete choice models aim at capturing the complex relations that link customers' individual tastes and socio-economic characteristics to their choices. Because of such complexity, discrete choice models are nonlinear and non-convex models. Consequently, merging demand modelling and supply optimization in the same framework requires a trade-off between complexity and computability.

Generally, demand-based optimization models can be modelled as Stackelberg games. Equivalent Stackelberg problems are common in transportation and in other markets when a single supplier or regulator knows the utility functions of its potential customers, who collectively play the follower role. From a mathematical modelling perspective, the result is an optimization problem having optimization problems in the constraints (Bracken and McGill, 1973), also known as bilevel program. An overview of bilevel optimization is provided by Colson *et al.* (2007).

In the literature, applications of demand-based optimization models include revenue management (Andersson, 1998, Talluri and Van Ryzin, 2004) and road tolling (Labbé *et al.,* 1998), among others. If we consider the works where discrete choice models are used to model demand, a majority of them propose nonlinear formulations and estimate customer choice probabilities with the multinomial logit model (MNL), whose advantage is the existence of a closed-form expression. However, MNL does not permit to consider random taste variation or correlation between alternatives. These limitations of the MNL led to the definition of more complex discrete choice models, such as the nested logit model (Ben-Akiva and Lerman, 1985), which relaxes the independence of irrelevant alternatives (IIA) assumption, and the mixed multinomial logit model, which can approximate any discrete choice model derived from random utility maximization under mild regularity conditions (McFadden and Train, 2000). A framework that can integrate any type of discrete choice model in a mixed integer linear program is introduced in Pacheco Paneque *et al.* (2017).

3 The modelling framework

3.1 Demand modelling

A market is considered where a number of different products are offered to a population. Customers are assumed to be utility maximizers who can only make a unitary and mutually exclusive purchase.

The notation is as follows. Let N represent the set of customers and let I indicate the set of choices available in the market. Utility functions U_{in} are defined for each customer $n \in N$ and alternative $i \in I$. Each utility function takes into account the socio-economic characteristics and the tastes of the individual as well as the attributes of the alternative. According to random utility theory (Manski, 1977), U_{in} can be decomposed into a systematic component V_{in} which includes all that is observed by the analyst and a random term ε_{in} which captures the uncertainties caused by unobserved attributes and unobserved taste variations. As a consequence, the resulting discrete choice models are naturally probabilistic. The probability that customer n chooses alternative i is defined as $P_{in} = \Pr[U_{in} = max_{j\in I} U_{jn}]$. In order to be able to estimate choice probabilities, assumptions must be made about the distribution of the error term. The two most used classes of discrete choice models, the probit and the logit, are built upon the assumption of normally distributed and Gumbel distributed error terms, respectively. For the sake of our discussion, it is worth pointing out that only logit models have a closed-form expression of the choice probabilities, which is nonlinear and non-convex. On the other hand, the choice probabilities of other models, including probit, must be expressed as integrals and approximated numerically, for instance by using simulation procedures (Train, 2009). As a result, while discrete choice models can accurately capture heterogeneous behavior on the demand side at a disaggregate level, their mathematical properties make it difficult to incorporate them in optimization models.

Pacheco Paneque *et al.* (2017) propose a linear formulation of the choice probabilities obtained by relying on simulation to draw from the distribution of the error term of the utility function. For each customer n and alternative i, a set R of draws are extracted from the known error term distribution, corresponding to different behavioral scenarios. For each scenario $r \in R$, the error term parameter ξ_{inr} is drawn and the utility becomes equal to $U_{inr} = V_{in} + \xi_{inr}$. Customers then deterministically choose the alternative with the highest utility, i.e. $P_{inr} = 1$ if $U_{inr} = max_{j\in I} U_{jnr}$ and $P_{inr} = 0$ otherwise. Over multiple scenarios, the probability that customer n chooses alternative i is equal to the number of times the alternative is chosen over the number of draws, i.e. $P_{in} = \frac{\sum_{r\in R} P_{inr}}{|R|}$. With a sufficient number of simulation draws, the obtained choice probabilities approximate the analytical formulation within a confidence interval.

3.2 Supply modelling

Suppliers are modelled as profit maximizers, according to the traditional microeconomic treatment and without loss of generality, but their goal could also be related to indicators other than profit. To optimize their objective function, suppliers must take strategic decisions about the availability of their products on the market and the corresponding attributes such as price and quantity. We assume that suppliers base their strategies on their knowledge of demand at a disaggregate level, which captures heterogeneity across the population, by incorporating discrete choice models into their optimization problem. The framework of the Stackelberg game models well the interaction between a supplier, acting as leader, and the customers, who represent the collective follower.

In addition to the notation introduced in §3.1, consider a supplier k participating in the market and let $I_k \subset I$ indicate the choices controlled by the supplier. Additionally, let $\mathcal{S}_k$ be the set of strategies that can be selected by the supplier. Each strategy $s \in \mathcal{S}_k$ is composed of a vector of decision variables, which we can separate into the vector p of all prices p_{in} for alternatives $i \in I_k$ and customers $n \in N$ and a generic vector X of all other decision variables. At this point, no assumption is made on the strategy set, which could be finite or infinite, or on the type of decision variables, which could be discrete or continuous.

For the sake of simplicity, the presented models assume that choice probabilities are estimated by using a multinomial logit model. The nonlinear version of the supplier's optimization problem can be written as follows:

$$\max_{s} \quad z_s = \sum_{i \in I_k} \sum_{n \in N} p_{in} P_{in} - \sum_{i \in I_k} c_i(X_i), \tag{1}$$

$$s.t. \quad P_{in} = \frac{\exp(V_{in})}{\sum_{j \in I} \exp(V_{jn})} \qquad \forall i \in I, \forall n \in N \tag{2}$$

$$V_{in} = \beta_{p,in} p_{in} + \beta_{in} X_{in} + q_{in} \qquad \forall i \in I, \forall n \in N. \tag{3}$$

The objective function (1) maximizes the profit of the supplier, calculated as the difference between the revenues obtained from the sales and the cost of offering the products. Notice that the function is generally non-convex due to the presence of the choice probabilities. Constraints (2) derive the choice probabilities. Constraints (3) define the deterministic utility functions, composed of an exogenous part q_{in} and an endogenous part which depends on the chosen strategy $s = (p, X)$, which links the upper-level problem with the lower-level problem.

For the linearized version of the model, we additionally define the auxiliary variables $U_{nr} = \max_i U_{inr}$, which capture the value of the highest utility for customer n in scenario r, while the binary decision variables P_{inr} identify the alternative i chosen by each customer n in each scenario r. Now constraints (2-3) can be written as follows:

$$\text{s.t.} \quad P_{in} = \frac{\sum_{r \in R} P_{inr}}{|R|} \qquad \forall i \in I, \forall n \in N \tag{4}$$

$$U_{inr} = \beta_{p,in} p_{in} + \beta_{in} X_{in} + q_{in} + \xi_{inr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{5}$$

$$U_{inr} \leq U_{nr} \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{6}$$

$$U_{nr} \leq U_{inr} + M_{U_{nr}}(1 - P_{inr}) \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{7}$$

$$\sum_{i \in I} P_{inr} = 1 \qquad \forall i \in I, \forall n \in N, \forall r \in R \tag{8}$$

$$P_{inr} \in \{0, 1\} \qquad \forall i \in I, \forall n \in N, \forall r \in R. \tag{9}$$

The utility functions (5) now include a drawn error term. Constraints (6-9) ensure that in each behavioral scenario customers deterministically choose the alternative yielding the highest utility.

3.3 Market modelling

Let us consider an oligopolistic market where we model demand as in §3.1 and supply as in §3.2. Due to oligopolistic market power, the payoff of each supplier is now a function of both the decisions of the customers and the strategies of the competitors. In other words, there are multiple suppliers that simultaneously solve a demand-based optimization problem. The result is a non-cooperative multi-leader-follower game in which each leader solves a best-response problem. In our framework, we search for pure strategy Nash equilibrium solutions of the static game.

The fixed-point iteration algorithm is a common approach to search for Nash equilibrium solutions in competitive markets. In transportation, examples include Fisk (1984) and Adler (2001), among others. Starting from an initial feasible solution to the problem, operators take turns in solving their best-response problem to the current market situation. Such sequential game terminates when a solution already reached in one of the previous iterations is repeated, as it would induce the same sequence of best responses as before. The solution of this game can be either a pure strategy Nash equilibrium for the game or a set of strategies for each player which would continue to be played cyclically. Modelling competition as a sequential game is attractive from a computational perspective, since the complexity of the problem is equivalent to the complexity of the Stackelberg game presented in §3.2. Furthermore, the sequential game is easily interpretable, since it reproduces the dynamic behavior of two or more players that do not know the competitors' objective function and are reactive to market changes.

However, the convergence proof of the algorithm depends on conditions such as having a convex payoff function, which are not verified in the multi-leader-follower game we want to solve. Consequently, by solving the problem as a sequential game there is no guarantee that a pure strategy Nash equilibrium exists or, if one is found, that it is unique. Moreover, different initial strategies could lead to different equilibria. All this means that in our research we can look at the fixed-point iteration algorithm as a heuristic approach rather than as an exact method.

3.3.1 A fixed-point MIP model

We propose a mixed integer programming model inspired by the fixed-point iteration algorithm. The key idea at the root of this model is to solve the sequential game as a one-step model by considering only two iterations of the fixed-point algorithm. We define as *distance* between two consecutive solutions a non-negative value measuring the difference in operators' decisions, in customers' decisions, or a mix of the two. If we start from an equilibrium solution of the problem, the distance between the initial solution and the next iteration's solution is equal to 0. On the other hand, if we do not start from an equilibrium solution, the distance is greater than 0, since at least one of the players changes its strategy.

To formalize, let K represent the set of suppliers, each controlling a subset of the alternatives that are available to the customers. We impose that $\cup_{k\in K} I_k \subset I$, in order not to have a captive market and allow customers to leave it without purchasing. Each supplier $k \in K$ has a set of strategies $\mathcal{S}_k$ from which to choose. We define the vector parameters p_s and X_s of the prices and of the other decisions of the supplier k playing strategy $s \in \mathcal{S}_k$. If we define as z_s the payoff obtained by supplier k when choosing strategy s, then in order to find a Nash equilibrium solution we need to verify that $z^*_s = z^{max}_k = \max_{s\in\mathcal{S}_k} z_s(s, s_{K\setminus\{k\}})\ \ \forall k \in K$. The binary decision variables x_s are equal to 1 if strategy $s \in \mathcal{S}_k$ is the best response of operator k to the initial configuration. Finally, the superscripts $'$ and $''$ are used to indicate the variables of the initial configuration and of the best response configurations, respectively.

Then, the MIP model with linearized choice probabilities can be written as follows:

$$\min \quad \sum_{k \in K} (|\boldsymbol{p}_k'' - \boldsymbol{p}_k'| + \alpha |\boldsymbol{X}_k'' - \boldsymbol{X}_k'|) \tag{10}$$

s.t. Initial configuration:

$$U_{inr}' = \beta_{p,in} p_{in}' + \beta_{in} X_{in}' + q_{in} + \xi_{inr} \quad \forall i \in I, \forall n \in N, \forall r \in R \tag{11}$$

$$U_{inr}' \leq U_{nr}' \quad \forall i \in I, \forall n \in N, \forall r \in R \tag{12}$$

$$U_{nr}' \leq U_{inr}' + M_{U_{nr}}(1 - P_{inr}') \quad \forall i \in I, \forall n \in N, \forall r \in R \tag{13}$$

$$\sum_{i \in I} P_{inr}' = 1 \quad \forall i \in I, \forall n \in N, \forall r \in R \tag{14}$$

Final configuration:

$$U_{inrs}'' = \beta_{p,in} p_{ins}'' + \beta_{in} X_{ins}'' + q_{in} + \xi_{inr} \quad \forall i \in I_k, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K \tag{15}$$

$$U_{inrs}'' = U_{inr}' \quad \forall i \in I \setminus I_k, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K \tag{16}$$

$$U_{inrs}'' \leq U_{nrs}'' \quad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K \tag{17}$$

$$U_{nrs}'' \leq U_{inrs}'' + M(1 - P_{inrs}'') \quad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K \tag{18}$$

$$\sum_{i \in I} P_{inrs}'' = 1 \quad \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K \tag{19}$$

$$P_{ins}'' = \frac{\sum_{r \in R} P_{inrs}''}{|R|} \quad \forall i \in I, \forall n \in N, \forall s \in \mathcal{S}_k \tag{20}$$

Best response constraints:

$$z_s = \sum_{i \in I_k} \sum_{n \in N} p_{ins} P_{ins}'' - \sum_{i \in I_k} c_i(X_{is}) \quad \forall s \in \mathcal{S}_k, \forall k \in K \tag{21}$$

$$z_s \leq z_k^{max} \quad \forall s \in \mathcal{S}_k, \forall k \in K \tag{22}$$

$$z_k^{max} \leq z_s + M(1 - x_s) \quad \forall s \in \mathcal{S}_k, \forall k \in K \tag{23}$$

$$\sum_{s \in \mathcal{S}_k} x_s = 1 \quad \forall k \in K \tag{24}$$

$$P_{inr}', P_{inrs}'' \in \{0, 1\} \quad \forall i \in I, \forall n \in N, \forall r \in R, \forall s \in \mathcal{S}_k, \forall k \in K \tag{25}$$

$$x_s \in \{0, 1\} \quad \forall s \in \mathcal{S}_k, \forall k \in K. \tag{26}$$

The objective function (10) minimizes the sum over all the suppliers of the distances between the final and the initial strategies. The absolute values can be linearized by expressing the arguments as the differences of two non-negative variables and by minimizing the sum of these variables in the objective function. Notice that different strategic decisions can have different units, therefore appropriate scale parameters must be defined when needed. Constraints (11-14) define the utilities and impose that customers choose the alternative with the highest utility in the initial configuration. Constraints (15-19) impose the utility maximization principle in the best response configurations. Here, utilities are evaluated for all strategies of all operators. In each strategic scenario, the decisions of the optimizing operator only affect the utility of its alternatives (15), while the utilities of the competitors' alternatives remain as in the initial configuration (16). Finally, constraints (20-24) state that operators always select the best response strategy to the initial configuration.

This MIP model is a one-step approach to find Nash equilibrium solutions for a competitive market by means of any MIP solver. Starting from an initial configuration, the model requires a number of strategic scenarios to be solved that is equal to $\sum_{k \in K} |S_k|$. Compared to the sequential game, it can also find near-equilibrium solutions, if no Nash equilibrium exists. Additionally, it enables discrimination between different equilibrium and near-equilibrium solutions by modifying the objective function. The same fixed-point model can incorporate nonlinear probabilistic customer choices. The description of the nonlinear case is omitted here. Numerical experiments that compare the two models are presented in §4.2.

4 Numerical experiments

This section discusses the results of some experiments aimed at understanding the properties of the models introduced in §3. The case study used for the tests is derived from Ibeas *et al.* (2014), where the choice of customers among three different parking alternatives is modelled with a mixed logit model. For the Stackelberg game we assume that two of these alternatives (paid underground parking and paid on-street parking) are managed by the same operator, while the third alternative (free on-street parking) is considered as the opt-out option, since it does not provide any revenue to the operator. For the multi-leader-follower game we assume that two of the three parking alternatives are managed by two different operators competing with each other, while the third alternative is kept as opt-out option. We further assume that for each alternative the operator decides on a unique price to be proposed to all customers. The test instances have the following size: 3 alternatives, 5-50 customers, 50-200 behavioral scenarios. All the proposed model are solved through the NEOS Server (Czyzyk *et al.,* 1998). MILP models are solved using CPLEX 12.7.0, while NLP and MINLP models are solved using Artelys Knitro 10.3.0.

4.1 The Stackelberg game

The nonlinear and the linear demand-based optimization models presented in §3.2 were tested on two discrete choice specifications, namely the multinomial logit model and the mixed logit model. The main difference between the two lies in the fact that the latter one allows for random taste variation by assuming a distribution for the taste coefficients. This implies that for the

Instance			MILP						NLP					
DCM	N	R	Time (s)	Obj	p_1	p_2	d_1	d_2	Time (s)	Obj	p_1	p_2	d_1	d_2
Logit	10	100	921	6.44	0.67	0.72	0.92	8.07	0.02	6.36	0.83	0.71	0	8.92
Logit	10	200	7027	6.43	0.66	0.72	0.99	8.05	0.02	6.36	0.83	0.71	0	8.92
Logit	50	50	7105	32.09	0.68	0.71	1.42	43.88	0.06	31.93	0.71	0.72	0.43	43.86
Logit	50	100	55020	32.19	0.68	0.73	2.80	41.66	0.06	31.93	0.71	0.72	0.43	43.86
Mixed	10	100	2378	5.38	0.55	0.63	2.72	6.18	0.05	5.31	0.55	0.63	2.79	6.03
Mixed	10	200	3942	5.21	0.54	0.61	2.94	5.95	0.29	5.22	0.56	0.64	2.96	5.60
Mixed	50	50	13285	27,33	0,58	0.67	13.80	29.08	0.45	27.20	0.58	0.66	13.64	29.06
Mixed	50	100	72000*	27,00*	0,56*	0.65*	13.92*	29.58*	0.70	26.92	0.56	0.66	14.79	28.39

Table 1: Numerical experiments on the Stackelberg game (uncapacitated)

Instance			MILP				MINLP			
DCM	N	R	Time (s)	Obj	p_1	p_2	Time (s)	Obj	p_1	p_2
Mixed	5	100	518	2.28	0.58	0.74	3	1.96	0.70	0.84
Mixed	5	200	4428	2.30	0.55	0.70	35	2.04	0.70	0.85
Mixed	10	100	4564	4.85	0.58	0.73	12	4.84	0.64	0.78
Mixed	10	200	72000*	4.70*	0.58*	0.68*	26	4.75	0.63	0.77
Mixed	50	50	72000*	26.09*	0.61*	0.77*	163	26.51	0.60	0.76
Mixed	50	100	72000*	25.71*	0.60*	0.74*	661	26.19	0.59	0.75

Table 2: Numerical experiments on the Stackelberg game (capacitated)

mixed logit it is necessary to simulate choice probabilites using Monte Carlo draws (Train, 2009).

Tables 1 and 2 show the results of the uncapacitated and of the capacitated instances, respectively. The latter ones include capacity constraints on the two operated alternatives, which require the use of binary variables that express whether an alternative is or is not available to a customer due to capacity limits. The experiments show that the nonlinear model converges much faster than the MILP model in all cases, and that computational times for the capacitated case are always higher than for the uncapacitated case. At this point we cannot draw conclusions on the effect of integrality constraints on the MINLP model, even though a sharp increase in computational time and the impossibility to converge to the optimal solution are expected when more integer variables are added to the model. The performance of the MILP model is primarily related to its combinatorial nature and to the weak formulation of the linear relaxation, which could be improved by adding valid inequalities.

Finally, we can observe that the choice probabilities and the demand shares of the MILP and NLP formulations are comparable in the three larger instances ($N = 50$) where both models converge, and this is confirmed by the experiments for the competitive case presented next. This indicates that using simulation to draw from the error term distribution leads to choice probabilities that approximate well those obtained with the probabilistic formulas.

Instance			MILP						MINLP					
DCM	N	R	Time (s)	Obj	p_1	p_2	d_1	d_2	Time (s)	Obj	p_1	p_2	d_1	d_2
Logit	5	50	68	0	0,05	0,15	1,28	3,72	78	0	0,05	0,15	1,54	3,46
Logit	5	100	203	0	0,05	0,15	1,65	3,35	78	0	0,05	0,15	1,54	3,46
Logit	5	200	818	0	0,05	0,15	1,55	3,45	78	0	0,05	0,15	1,54	3,46
Logit	10	50	208	0	0,05	0,15	2,74	7,26	94	0	0,05	0,15	2,85	7,15
Logit	10	100	3679	0	0,05	0,15	2,86	7,14	94	0	0,05	0,15	2,85	7,15
Logit	10	200	5595	0	0,05	0,15	2,84	7,16	94	0	0,05	0,15	2,85	7,15
Logit	50	25	6894	0	0,05	0,15	11,20	38,80	1151	0	0,05	0,15	10,72	39,29
Logit	50	50	16400	0	0,05	0,15	10,60	39,40	1151	0	0,05	0,15	10,72	39,29
Logit	50	100	6124	0	0,05	0,15	10,81	39,19	1151	0	0,05	0,15	10,72	39,29
Mixed	5	50	70	0	0,10	0,20	1,96	3,04	849	0	0,10	0,20	2,05	2,95
Mixed	5	100	170	0	0,15	0,20	1,52	3,48	747	0	0,10	0,20	2,22	2,78
Mixed	5	200	1013	0	0,10	0,20	2,13	2,87	2962	0	0,10	0,20	2,07	2,93
Mixed	10	50	291	0	0,15	0,25	4,16	5,84	2019*	0,09*	0,30*	0,39*	3,95*	6,05*
Mixed	10	100	2204	0	0,15	0,25	3,84	6,16	3499	0	0,10	0,20	3,92	6,08
Mixed	10	200	3589	0	0,10	0,20	4,17	5,83	4413	0	0,10	0,20	4,18	5,82
Mixed	50	25	985	0	0,10	0,20	17,24	32,76	7035	0	0,10	0,20	17,09	32,91
Mixed	50	50	13923	0	0,15	0,25	18,28	31,72	16242*	0,19*	0,13*	0,32*	31,42*	18,58*
Mixed	50	100	28682	0	0,15	0,25	18,31	31,69	36000*	-	-	-	-	-

Table 3: Numerical experiments on the fixed-point MIP model

Instance				MILP				MINLP			
DCM	N	R	$\|S_k\|$	Time (s)	Obj	p_1	p_2	Time (s)	Obj	p_1	p_2
Logit	10	100	11	3679	0	0,05	0,15	94	0	0,05	0,15
Logit	10	100	21	16524	0	0,02	0,10	194	0	0,02	0,10
Logit	10	100	31	59096	0	0,02	0,11	719	0	0,02	0,11
Mixed	10	100	11	2204	0	0,15	0,25	3499	0	0,10	0,20
Mixed	10	100	21	4023	0	0,12	0,22	11006*	0,05*	0,19*	0,26*
Mixed	10	100	31	5017	0	0,12	0,21	19401*	0,02*	0,11*	0,19*

Table 4: Numerical experiments to test the effect of the strategy set size

4.2 The multi-leader-follower game

For the competitive case, we report the numerical experiments performed using the fixed-point optimization model outlined in §3.3.1. Table 3 shows that in the case of a logit formulation the nonlinear model converges faster to optimality, as there is no need for simulation. On the other hand, when using a mixed logit formulation, the linear model generally outperforms the nonlinear model, which fails to converge on larger instances. Compared to the results of the Stackelberg game, the substantial worsening of the computational performance of the nonlinear model can be imputed to the discretized price parameters and to the binary decision variables of the upper-level problems, while the relatively good performance of the MILP model can be explained by the reduction of the solution space due to the limited set of response strategies. In particular, it can be seen that the linear model, which is structured around a simulation framework, has similar computational performances on the logit and the mixed logit model. The latter finding is particularly encouraging, because it indicates that the MILP formulation for the demand-based optimization model could potentially embed the most complex and accurate discrete choice models. Finally, Table 4 shows that, as expected, computational times are influenced by the size of the players' strategy set.

4.3 Discussion

The numerical experiments performed so far indicate that the main factors affecting the computational performance of the different models include the form of the choice probabilities (nonlinear or linearized) at the demand level and the type of decision variables (continuous or discrete) at the supply level. On one side, the nonlinear formulation is non-convex. This means that it can be efficient when all the supply decision variables are continuous and the problem is well-conditioned, but it becomes intractable when many discrete variables are introduced, since convergence cannot be proved due to local optima. On the other side, the linear formulation is convex, but the combinatorial nature of the simulation framework makes it unfit to solve realistic multi-leader-follower games.

The fixed-point MIP model presented in §3.3.1 has shown promising results, but its large scale applicability depends on the quality of the strategy set generation techniques that are employed to select potential equilibrium solutions and best-response strategies for all competitors. In the numerical experiments, an arbitrary price discretization has been applied, but the inclusion of capacity variables, price discrimination across customers or multiple alternatives for each supplier would make the strategy sets of the original problem grow exponentially. For this reason, the fixed-point MIP model could constitute one of the final blocks of an algorithmic approach, using as input the candidate solutions found in the earlier stages. The model's ability to discriminate between different equilibrium or near-equilibrium solutions could then be exploited, also at an applied level, to test different objective functions and competitive behaviors.

5 Conclusions and future research

In this paper, we presented a demand-based optimization framework to model competition in oligopolistic markets. We described our methodological approach which includes three components: (i) demand, for which discrete choice models are used to take into account preference heterogeneity and to model individual decisions according to the utility maximization principle; (ii) supply, for which a bilevel optimization problem models how strategic decisions are affected by the knowledge of the customers' utility functions; (iii) market, for which the interactions caused by the existence of competitors with conflicting interests are modelled as a non-cooperative game to understand market equilibrium solutions.

Modelling demand at a disaggregate level implies that the resulting supply payoff functions are generally non-convex. As a consequence, it is not possible to prove existence and uniqueness of Nash equilibria, nor to solely rely on derivative-based methods. We proposed a mixed integer programming formulation based on the fixed-point iteration algorithm, which can be applied to discrete games. Numerical experiments show that the MIP model can solve small-size instances to optimality. Its applicability to larger problems depends on the quality of the strategy set generation techniques that are employed to select potential equilibrium solutions and best-response strategies for all competitors.

Our research is currently investigating an algorithmic approach to find equilibrium or near-equilibrium solutions in oligopolistic markets. First, a quick heuristic such as the sequential game with nonlinear choice probabilities and with multiple restarts could be used to explore the solution space and identify potential equilibrium regions, as done in Adler (2001). If one or more pure strategy Nash equilibrium solutions are found, then they are market equilibria for the problem. Else, if a cyclic equilibrium is detected, none of the combinations of the suppliers' strategies yields an equilibrium solution. However, the strategies retrieved in the cyclic equilibria could be used as restricted set of candidate best-response strategies in the fixed-point optimization model. Then, the fixed-point optimization model could be used to find subgame equilibria on the restricted strategy sets and to evaluate the deviation from equilibrium of the different solutions. It is reasonable to believe that suppliers are unwilling to change their strategies for a marginal increase in profits, if this led to a price war that would in turn affect their later profits at a greater extent, falling into a prisoner's dilemma-like pattern. To address this point, we plan to look further into the economic literature on epsilon-equilibria and tacit collusion, e.g. Radner (1980) and related research, which could enhance the applicability of our framework to real-life markets, where perfect profit maximization behaviors and pure strategy equilibria are unlikely to exist.

Later, our algorithmic approach will be used to study an oligopolistic market within the transport industry. Some of the challenges to explore include: (i) at a demand level, the use of more advanced discrete choice models, such as nested logit or latent class models, and the inclusion of endogenous variables other than price in the utility function; (ii) at a supply level, the possibility to offer different products or prices to different customers; (iii) at a market level, the definition of near-equilibrium regions for real life applications.

6 References

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